Duality Theory
for
Optimal Mechanism Design

Yiannis Giannakopoulos
St Anne's College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy
Hilary 2015

This work is dedicated to
the memory of my father Αλέξη
for teaching me how to be a *free* man

"...οὐκ Ἀθηναῖος οὐδὲ Ἕλλην, ἀλλὰ Κόσμιος..."
Socrates in Plutarch's *Moralia – De Exilio*

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Abstract

In this work we present a general duality-theory framework for revenue maximization in additive Bayesian auctions involving multiple items and many bidders whose values for the goods follow arbitrary continuous joint distributions over some multi-dimensional real interval. Although the single-item case has been resolved in a very elegant way by the seminal work of Myerson [1981], optimal solutions involving more items still remain elusive. The framework extends linear programming duality and complementarity to constraints with partial derivatives. The dual system reveals the natural geometric nature of the problem and highlights its connection with the theory of bipartite graph matchings.

We demonstrate the power of the framework by applying it to various special monopoly settings where a seller of multiple heterogeneous goods faces a buyer with independent item values drawn from various distributions of interest, to design both exact and approximately optimal selling mechanisms. Previous optimal solutions were only known for up to two and three goods, and a very limited range of distributional priors. The duality framework is used not only for proving optimality, but perhaps more importantly, for deriving the optimal mechanisms themselves.

Some of our main results include: the proposal of a simple deterministic mechanism, which we call Straight-Jacket Auction (SJA) and is defined in a greedy, recursive way through natural geometric constraints, for many uniformly distributed goods, where exact optimality is proven for up to six items and general optimality is conjectured; a scheme of sufficient conditions for exact optimality for two-good settings and general independent distributions; a technique for upper-bounding the optimal revenue for arbitrarily many goods, with an application to uniform and exponential priors; and the proof that offering deterministically all items in a single full bundle is the optimal way of selling multiple exponentially i.i.d. items.

Acknowledgements

The sense of fulfilment I get as I am writing the last few words of this thesis can only be matched by the sense of gratitude I am feeling towards Elias for everything he has offered me through the years. He showed me how research needs to have vision, but most importantly integrity and virtue. His insistence in asking only the most important and beautiful of questions, his unique clarity of thought, and his holistic, passionate approach to knowledge and learning made working with him not just exciting and deeply satisfying, but above all fun. He supported me through various difficulties and obstacles, by trusting and believing in me. I will always remember our discussions over tea, coffee, wine, and the occasional pint. Elia, thank you for this wonderful journey! You've been a great teacher and a great friend.

There is not a single more important person in my life than my sister Angeliki. We've been through so many things together, that I cannot possibly imagine this world without her. I am indebted to her for the countless happy memories, all the unconditional support and of course her understanding of the many difficult and annoying aspects of my character. A big thanks also to my little sister Maria for how proud she's been making me.

Firstly, I want to deeply thank some very special people that have made life so much more beautiful for me during the last few years: Penny, Aloe, Michaela, Sevgi and Christina. I've learned many things from you, and you have shaped my character and perception of the world in fundamental ways that you cannot possibly conceive. In the words of Nikos Kazantzakis:

> *I felt once more how simple and frugal a thing is happiness: a glass of wine, a roast chestnut, a wretched little brazier, the sound of the sea. Nothing else.*

A big thank you to all my amazing friends in Greece for the great times and for giving me a purpose for coming back home: Angeliki, Giorgos, Alexandra, Thodoris, Yiannis V., Irene, Yiannis R., Spyros, Virginia, Stevie and Dimitris L.. But above all the other two "ημίσκουμπρες" Kostas and Nick, for all the experiences we have shared together and for being (almost) as ridiculous as I am. Oxford is a wonderful, exciting place and I had the opportunity to meet here some of the most interesting people I've ever known. Some of them I had the privilege to call them my friends, and I thank them for making my life here fun: Hoi Lun, Rhyannon, Lindsey, Alicia, Andreas G., Matthias, Leanne, Kristen, Ben, Caitlin, Natalie, Hillary, Mat, Joanna, Hannah, Henriette and Katie; Arturo and Rupert for all the intense discussions.

I also want to thank my wine-tasting buddies from back home, Irene, Giorgos D., Michael, Agis, Danis and Yiannis R. for starting this wonderful journey together, and all the members of the Oxford Blind-Tasting Society for sharing their passion and knowledge of wine; I will always regret putting maths and research as a first priority and not dedicating the required amount of time to help with the varsity team. Thanks to all the other members of the Algorithmic Game Theory group in Oxford CS: Maria K., Matthias, Jie and Philip for creating a stimulating and welcoming environment; and to Michalis S., Panagiotis L. and Katia from Athens. My academic brother, Giorgos Christodoulou, has offered me valuable advice throughout my studies. I also thank Dave R. for the obligatory British wisdom, and Andreas G. for refusing to grow up; just like me.

Finally, I am grateful to the following institutions for their generous financial support during my graduate studies: the Computer Science department of the University of Oxford, the Engineering and Physical Sciences Research Council (EPSRC), the European Union through FP7-ICT grant 284731 (UaESMC) and ERC Advanced Grant 321171 (ALGAME), and Propondis Foundation.

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Chapter 1

Introduction

1.1 Games, Mechanisms and Auctions

Consider a monopolist of a single good facing multiple buyers. What selling mechanism should he deploy in order to maximize his[1] revenue? A natural choice, for instance, would be to ask from the buyers to submit bids, and then just allocate the item to the player with the highest offer, for a payment equal to her bid. This is known as a first-price auction. Notice that there is an underlying *game* taking place here: each buyer knows how much the item is worth to her, and the available *strategies* are what bid to submit; being rational and selfish, she might lie about her true value for the item if this is to increase her own personal gain. For example, assume two buyers, who value the good £10 and £20, respectively. Think of this as the “amount of happiness” they receive in case they win the item, called *valuation*. If players were to be completely honest, the seller could extract a full revenue of £20 (from the first buyer). However, in such an auction the first player would have a motive to lie, declare a lower bid of £10.05, still get the item and at the same time also manage to strictly reduce the payment she has to submit. This kind of behaviour can cause a high degree of complexity both in analyzing, as well as implementing, a stable equilibrium state for the system, due to the various interweaving and contradicting preferences and reactions of the players. So, we would like to design auctions which can make sure players are *truthful*, that is they have no incentive to misreport their private preferences. This is essentially the subject of the area known as *mechanism design*. For example, a truthful auction here would be to allocate the item to the highest-bidding buyer but only charge her the *second-highest* bid, known as a second-price or *Vickrey auction*. It is simple to verify that, by honestly reporting a bid equal to her own private value for the item, each buyer maximizes her *utility*, i.e. her valuation minus the payment, no matter what the other player chooses to do. Put it in game-theoretic terms, truth-telling is a *dominant*

[1] Without implying any social correlation whatsoever, and for reasons of simplicity and clarity, throughout this thesis we choose to use masculine pronouns when referring to the seller and feminine when referring to the bidders of an auction.

strategy for all players.

The auctioneer has some prior, *incomplete* knowledge of the buyers' private values for the goods, in the form of a joint probability distribution. The goal then is to design an auction that maximizes the *expected revenue* of the seller, that is the total payment received from the buyers, under that distribution. Let's take a moment here to contemplate on the vastness of available options. The auctions to consider can be deterministic, like the first- and second-price ones given above, but also randomized *lotteries*: for example, charging the players a certain amount in exchange for a chance of winning the item with a given probability; they can also be *indirect*, unlike the ones we've seen so far, meaning that instead of asking the players for a straightforward revelation of their values in the form of bids, they may demonstrate a much more complex structure: multiple rounds, involving interaction between the designer and the players through a general form of messages/actions; even truthfulness can be challenged: it is without doubt a desirable feature, but on the other hand we may be willing to make compromises if this is to increase our profits.

Fortunately, due to the celebrated *Revelation Principle*, it can be shown that we can restrict attention just in direct-revelation, truthful mechanisms, without a loss for the revenue maximization objective. This is a great help in our quest for optimal auction design, since it reduces the optimization space significantly, both in terms of size but also in complexity of structure. As a matter of fact, for the specific example of a single good with many buyers and values drawn from independent distributions, Myerson showed in his seminal work [58] that a simple deterministic mechanism, a simple twist to the standard second-price auction, can achieve exact optimality: just add a *reserve price*, i.e. a threshold under which no buyer can win the item. Equivalently, this can be seen as introducing a "dummy" player in the auction with a bid equal to this price. Myerson also provides an elegant, closed-form formula for determining that threshold.

Disappointingly enough though, more than 30 years after that work, *optimal auctions for more than just a single item remained completely elusive.* Despite the efforts of economists, and more recently computer scientists as well, the quest for generalizing Myerson's results in multidimensional settings has been unsuccessful: we have a very poor understanding of the structure of revenue-maximizing auctions, even in the case of a single buyer and just two or three goods.

Our goal in this thesis is to take a step forward towards that direction: present a new way of attacking the problem through the development of a novel, general duality-theory framework that can be then applied in various special cases to give multiple new results, as well as provide us with a deeper and more clear understanding of the structure and characteristics of multidimensional optimal auctions.

1.2 Outline of the Thesis and Main Results

The building block for the majority of the results in this work is a *new duality-theory framework for revenue maximization in settings involving multiple items* and many additive-valuation buyers, whose values for the goods are drawn from an arbitrary continuous joint distribution over some multidimensional real interval. We stress here that we do *not* impose a requirement for determinism but optimization is done with respect to the wider class of all feasible randomized auctions, i.e. we allow for lotteries.

This framework is developed in its full generality in Chapter 3 and its critical components, namely weak duality and (approximate) complementarity, are formally proved. When designing these tools, our priority was to make them resemble as much as possible their counterparts from classic linear programming theory, both in form but also in usage. As a result, we believe the reader will immediately feel familiar with the formulations and tools, which involve simple, closed-form expressions that can readily provide manageable formulas when the particular distributional priors are plugged-in. Furthermore, this can result in a more intimate understanding of the underlying mechanics, and better and more natural intuition with respect to the results. Many fine points and aspects of the framework are also discussed in Section 3.3.

A critical feature that assists this duality approach is the formulation of the revenue maximization problem as a functional optimization one, with respect to the utility functions of the players. This analytic approach is achieved by a well-known characterization of truthfulness through these functions' derivatives. We give the necessary background and fix our environment and notation in Chapter 2. As a warm-up, in Section 2.3.3 we also present some very simple *approximation* results.

The remainder of the thesis is devoted to demonstrating the use of the duality framework by applying it to various open problems in multidimensional optimal auctions. In particular, we will restrict our attention to single-buyer settings with independently (but not necessarily identically) distributed item values. Previous related work is extensively discussed in Section 2.4.

First, in Chapter 4 we deal with the "canonical" open problem in the area, that of maximizing the revenue of a monopolist of multiple heterogeneous goods facing a buyer whose values for the items are *uniformly* distributed. We propose a deterministic selling mechanism, called Straight Jacket Auction (SJA), prove its optimality for up to six items and conjecture this holds generally. Duality is used not only for the proof of optimality, but perhaps more importantly, in the design of SJA itself, which leads to a novel understanding of the particular setting: interesting notions from geometry and the theory of bipartite graph matchings come into play, and the definition of SJA demonstrates a surprising recursive character.

We then focus on *two-item* settings. This restriction on the number of goods will allow us a much greater degree of generality and abstraction with respect to the valuation priors. In Chapter 5 we provide sufficient conditions for revenue maximization in a two-good monopoly where the buyer's valuations for the items come from independent (but not necessarily identical) distributions over bounded intervals. Under certain distributional assumptions, we give exact, closed-form formulas for the prices and allocation rules of the optimal selling mechanisms. As a side result we give the first example of an optimal mechanism in an i.i.d. setting over a support of the form $[0, b]$ which is *not* deterministic. Since our framework is based on duality techniques, we were also able to demonstrate how slightly relaxed versions of it can still be used to design mechanisms that have very good approximation ratios with respect to the optimal revenue, through a *"convexification"* process.

For the special case of uniform distributions, in Section 5.6 we provide a new, much simplified and straightforward proof to a result of Pavlov [66] regarding the revenue maximizing mechanism for selling two goods with uniformly i.i.d. valuations over intervals $[c, c+1]$, to an additive buyer. This is done by explicitly constructing optimal dual solutions to a relaxed version of the problem, where the convexity requirement for the bidder's utility has been dropped. Their optimality follows directly from their structure, through the use of exact complementarity. For $c = 0$ and $c \geq 0.092$ it turns out that the corresponding optimal primal solution is a feasible selling mechanism, thus the initial relaxation comes without a loss, and revenue maximality follows. However, for $0 < c < 0.092$ that's not the case, providing the first clear example where relaxing convexity provably does not come for free, even in a two-item regularly i.i.d. setting.

Next, in Chapter 6 we turn our attention to *approximation* techniques when an *arbitrary number of items* are involved. Using the duality theory framework we derive simple, closed-form formulas for bounding the optimal revenue of a monopolist selling many heterogeneous goods, in the case where the buyer's valuations for the items come i.i.d. from a uniform distribution and in the case where they follow independent (but not necessarily identical) exponential distributions. We apply this in order to get in both these settings specific performance guarantees, as functions of the number of items m, for the simple deterministic selling mechanisms studied by Hart and Nisan [38], namely the one that sells the items separately and the one that offers them all in a single bundle.

In Section 6.3.2 we also propose and study the performance of a natural randomized mechanism for exponential valuations, called Proportional. As an interesting corollary, for the special case where the exponential distributions are also identical, we can derive that offering the goods in a single full bundle is the optimal selling mechanism for *any number of items*. To our knowledge, this is the first result of its kind: finding a revenue-maximizing auction in an additive setting with arbitrarily many goods.

Finally, there are a couple of topics that, although we found to be interesting on their own, especially for possible future reference by the interested researcher, we choose to present them in the appendix so as not to interrupt the dynamic of the main text.

In Appendices A.2 and A.4 we give alternative, explicitly constructive dual solutions for the case of two uniformly and exponentially distributed goods, respectively, and in Appendix A.3 we provide an abstraction of an upper-bound technique we used in Section 6.1 for the special case of uniform distributions to general distributions, that involves the extension of the traditional Myersonian [58] notions of virtual valuations and regularity.

1.3 List of Papers

The majority of our results in this thesis are from a series of papers, listed below for completeness. Those in Appendices A.2 to A.4 as well as the discussion in Section 2.3.3, however, appear here for the first time.

[1] Yiannis Giannakopoulos and Elias Koutsoupias
Selling Two Goods Optimally
In *Proceedings of 42nd International Colloquium on Automata, Languages, and Programming (ICALP'15)*, July 2015.
(Best paper award)

[2] Yiannis Giannakopoulos
Bounding the Optimal Revenue of Selling Multiple Goods
In *Theoretical Computer Science,* 581: 83–96, 2015.

[3] Yiannis Giannakopoulos
A Note on Selling Optimally Two Uniformly Distributed Goods
In *CoRR: abs/1409.6925*, September 2014.

[4] Yiannis Giannakopoulos and Elias Koutsoupias
Duality and Optimality of Auctions for Uniform Distributions
In *Proceedings of the 15th ACM Conference on Economics and Computation (EC'14)*, pp. 259–276, June 2014.
Full version in *CoRR: abs/1404.2329*.

Chapter 2

Fundamentals of Auction Theory

In this chapter we formalize the notions intuitively discussed in the introductory Chapter 1. We will give a crash course in the fundamentals of Auction Theory, that are needed in order to develop our main results in the following chapters. We do that, by providing a somehow more abstract path through Mechanism Design; we believe this more rigorous approach is essential for the deeper understanding of the mechanics, but even more importantly, of the powerful foundations and potential of auction design. However, in no way this is a complete, or even balanced, introduction to the subject, something that would be infeasible for the scope of this thesis; the interested reader is encouraged to look at some of the excellent textbooks and surveys related to the subject, e.g. [46, Part A], [45], [44, Chapter 9] and [47].

2.1 Notation

Let us now fix some initial notation that will be used throughout the rest of the thesis. For any positive integer m, we denote $[m] = \{1, 2, \dots, m\}$. The reals are denoted by $\mathbb{R}$, their nonnegative subset by $\mathbb{R}_+$ and the unit interval by $I = [0, 1]$. In general, we will use a bold typeface for matrices and vectors, and a normal-weight type for their single-dimensional components, e.g. $\mathbf{x} = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$ is an m-dimensional real vector. Let $\mathbf{0}_m = (0, 0, \dots, 0)$ and $\mathbf{1}_m = (1, 1, \dots, 1)$ denote the m-dimensional zero and unit vectors, respectively. We will drop subscript m whenever this causes no confusion. For two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ we write $\mathbf{x} \leq \mathbf{y}$ as a shortcut to $x_j \leq y_j$ for all $j \in [m]$. Inner vector product will be denoted by standard dot notation $\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^{m} x_j y_j$. For any matrix $\mathbf{x} \in \mathbb{R}^{n \times m}$, $\mathbf{x}_i$ will denote its i-th (m-dimensional) row vector.

It is standard, and very convenient, in game-theoretic treatments to use $\mathbf{x}_{-j}$ for the vector we are left with if we remove the j-th coordinate from $\mathbf{x}$, i.e.

$$\mathbf{x}_{-j} = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n).$$

Using that, one can have $(y, \mathbf{x}_{-j})$ to be the result of replacing the i-th component of $\mathbf{x}$ with a new value y. In a similar way, if $\mathbf{x} \in \mathbb{R}^{n\times m}$ is a matrix, then we can change the value at the i-th row and j-th column element x_{ij} using $(y, \mathbf{x}_{-i,j})$.

For a real function $g : \mathbb{R}^m \to \mathbb{R}$ we use standard notation $\nabla g(\mathbf{x}) = \left(\frac{\partial g(\mathbf{x})}{\partial x_1}, \frac{\partial g(\mathbf{x})}{\partial x_2}, \dots, \frac{\partial g(\mathbf{x})}{\partial x_m}\right)$ and extending this to functions $g : \mathbb{R}^{n\times m} \to \mathbb{R}$ over matrices, for $i \in [n]$ we will denote

$$\nabla_i\, g(\mathbf{x}) \equiv \left(\frac{\partial g(\mathbf{x})}{\partial x_{i,1}}, \frac{\partial g(\mathbf{x})}{\partial x_{i,2}}, \dots, \frac{\partial g(\mathbf{x})}{\partial x_{i,m}}\right).$$

Notice how only the derivatives with respect to the variables in row $\mathbf{x}_i$ appear.

For probability distributions, we will use upper-case symbols like F to denote their (cumulative) distribution function and lower-case f for their density. If X is a random variable, then $X^{(i:n)}$ will denote the i-th order statistic of X, i.e. the i-th smallest out of n independent draws from X: $X^{(1:n)} \leq X^{(2:n)} \leq \dots \leq X^{(n:n)}$. We will sometimes use the shortcut $\mathcal{U}$ for the uniform distribution over I, that is distribution $F(x) = x$, and $\mathcal{E}(\lambda)$ for the exponential distribution over $\mathbb{R}_+$ with parameter $\lambda > 0$, i.e. $F(x) = 1 - e^{-\lambda x}$. For simplicity, we will also use $\mathcal{E} \equiv \mathcal{E}(1)$. Harmonic numbers are denoted by $H_n = \sum_{i=1}^{n} \frac{1}{i}$.

Finally, for some set-theoretic notation, we use B^A for the family of all functions from set A to B, and $A \times B$ for the Cartesian product of A and B. We denote $\prod_{j=1}^{m} A_i = A_1 \times A_2 \times \dots \times A_m$, and deploying the game-theoretic notation we introduced above, if $A = \prod_{j=1}^{m} A_j$ then for any $j \in [m]$: $A_{-j} = \prod_{k=1}^{j-1} A_k \times \prod_{k=j+1}^{m} A_k$. We will denote the standard Lebesgue measure over Euclidean spaces with μ, and we will say that a certain property holds *almost everywhere (a.e.)* if it is true except from a set of zero measure.

2.2 Mechanism Design

2.2.1 Games

Any serious approach to the design and analysis of auctions goes through their modelling as games of incomplete information, and thus some minimal amount of game-theoretic terminology would be needed for introducing the key Mechanism Design notions in Section 2.2.2.

You can think of a *game* $\mathcal{G}$ as a structure consisting of a finite set of *players* $[n]$, and for each player $i \in [n]$ a set of *strategies* (or *actions*) S_i available to her together with a *utility* (or *payoff*) function $u_i : \prod_{i=1}^{n} S_i \longrightarrow \mathbb{R}$ over them. The intuitive interpretation is that, every possible *strategy profile* (or *outcome*) $\mathbf{s} = (s_1, s_2, \dots, s_n) \in S = \prod_i S_i$ of $\mathcal{G}$ results in an amount of "happiness" measured by $u_i(\mathbf{s})$ for each player i, and thus, being fully rational and selfish, each one of them will try to choose a strategy s_i that will maximize her own payoff.

The most robust *solution* concept (or *equilibrium*) of such a game would be one at which each player has a clear optimal strategy, that maximizes her personal gain no matter what the other players would choose to do:

Definition 2.1 (Dominant strategies)**.** Let $\mathcal{G} = (\{S_i\}_{i\in[n]}, \{u_i\})$ be a game. A strategy profile $\mathbf{s}^*$ is a *dominant strategy equilibrium* of $\mathcal{G}$ if, for every player $i \in [n]$, s_i^* is a *dominant strategy* for i, i.e.

$$u_i(s_i^*, \mathbf{s}_{-i}) \geq u_i(s_i, \mathbf{s}_{-i}) \qquad \text{for all } \; s_i \in S_i, \; \mathbf{s}_{-i} \in S_{-i}.$$

Notice here that not all games have dominant strategy equilibria (see e.g. the "Battle of the Sexes" game), and thus weaker solution concepts with guaranteed existence have been developed for the analysis of players' behaviour, most notably the celebrated *Nash equilibrium* [61].

Bayesian games Many times we want to model strategic interactions between agents when full knowledge of their private preferences among all of them is a highly unrealistic assumption to make. For instance, a player usually will know her own utility function but only have some *incomplete information* about the utilities of the other participants, given by probability distribution. Then, she has to base her strategic analysis and behaviour on *expectation* over her prior beliefs for the preferences of the others.

To formalize this, we define an *incomplete information* (or *Bayesian*) *game* to consist, for every player $i \in [n]$, of a set of *types* T_i, a set of *actions* A_i and a *utility function* $u_i : T_i \times \prod_i A_i \longrightarrow \mathbb{R}$. The *strategy* set of a player i then is a rule that relates her possible types to actions: $S_i = A_i^{T_i}$. There is also a probability distribution F over the space of possible type profiles $T = \prod_{i=1}^n T_i$, which is common knowledge among the players.

An intuitive way of interpreting this definition is the following: each type t_i is a parameter that determines how the preferences of player i would look like, what "kind of agent" she would be. Think of this as something beyond the control of the players, which is going to be determined by "nature". Then, a type profile $\mathbf{t}$ is drawn from distribution F and a *signal* with the actual value of t_i is sent to each player i (but not to the other players). Now, there is essentially no layer of uncertainty any more regarding the payoffs, and a full information game $(\{A_i\}, \{\tilde{u}_i\})$ could be played over the possible actions $a_i \equiv s_i(t_i)$ of the players, where $\tilde{u}_i(\mathbf{a}) \equiv u_i(t_i; \mathbf{a})$ for all $\mathbf{a} \in \prod_i A_i$. But there is a catch: since player i is unaware of what types $\mathbf{t}_{-i}$ nature actually chose the other players to be, she has to base her game-playing analysis on expectation with respect to distribution F, conditioned of course upon her own realized value t_i. Put it another way, from player's i perspective, she faces a distribution over a whole family of possible full-information games.

The notion of a dominant strategy equilibrium can be naturally generalized to

Bayesian games, to denote an situation where each player has an optimal plan of action, no matter how the other players decide to act and no matter the realized state of nature $\mathbf{t} \sim F$. Such a solution concept is extremely powerful: if it exists, then it definitely captures the expected outcome of the game, assuming the players demonstrate rational behaviour:

Definition 2.2 (Dominant strategies in Bayesian games)**.** Let $\mathcal{G} = (\{T_i\}_{i\in[n]}, \{A_i\}, \{u_i\}, F)$ be an incomplete information game. A strategy profile $\mathbf{s}^*$ is a *dominant strategy equilibrium* of $\mathcal{G}$ if, for every player $i \in [n]$, s_i^* is a dominant strategy for i, i.e.

$$u_i(t_i; (s_i^*(t_i), \mathbf{a}_{-i})) \geq u_i(t_i; (a_i, \mathbf{a}_{-i})) \qquad \text{for all } \ t_i \in T_i \ \text{ and } \ a_i \in A_i, \ \mathbf{a}_{-i} \in A_{-i}.$$

However robust and desirable such an equilibrium might be, it essentially overrides a priori the Bayesian layer of incomplete information: notice how distribution F is not used anywhere in Definition 2.2. Thus, the following provides a weaker, but still natural solution concept for Bayesian games:

Definition 2.3 (Bayes-Nash equilibrium)**.** Let $\mathcal{G} = (\{T_i\}_{i\in[n]}, \{A_i\}, \{u_i\}, F)$ be an incomplete information game. A strategy profile $\mathbf{s}^*$ is a *Bayes-Nash equilibrium* of $\mathcal{G}$ if no player i has an incentive to unilaterally change her strategy, under her prior knowledge of nature. Formally, for every $i \in [n]$,

$$\mathbb{E}_{\mathbf{t}_{-i}\sim F_{-i}|_{t_i}}\left[u_i(t_i; (s_i^*(t_i), \mathbf{s}_{-i}^*(\mathbf{t}_{-i})))\right] \geq \mathbb{E}_{\mathbf{t}_{-i}\sim F_{-i}|_{t_i}}\left[u_i(t_i; (a_i, \mathbf{s}_{-i}^*(\mathbf{t}_{-i})))\right]$$

for all $t_i \in T_i$ and $a_i \in A_i$, where $F_{-i}|_{-t_i}$ denotes the conditional distribution of F upon its i-th component being realized at t_i.

For the rest of the thesis, we make the deliberate choice to use *dominant* strategies as the default solution concept for our analysis, due to its robustness and clarity. Also, since it is the strongest notion, all results can be translated to the realm of Bayes-Nash equilibria in a straightforward way. So, except stated otherwise, *equilibrium* from now on will refer to *dominant strategy equilibrium*.

For a serious treatment of the subject of Game Theory we recommend any of the standard textbooks [28, 65, 59].

2.2.2 Mechanisms

Although games seem to capture the fundamental notion of strategic interaction between players, the model so far is completely *passive*: the system somehow reaches a stable state, an equilibrium point, which is an internal property of the game and we can have no control over it at all. But for the auction problems we are interested in studying, there is some external designer who wants to create protocols that implement some desired objective, for example maximization of the seller's revenue or minimization of the total social cost. This means that we would like to have a way of designing *rules* for the games in a way that the previous, passive convergence to a stable state will coincide with an outcome that satisfies our design needs. This is essentially the subject matter of the entire area of Mechanism Design, which for that reason sometimes is being referred to as *reverse game theory*.

In general, a *Bayesian mechanism design environment* comprises of a finite set of players $[n]$, where player i is of type $t_i \in T_i$ and, as discussed in the previous Section 2.2.1, the type profile $\mathbf{t}$ will be instantiated by "nature" according to some distribution F which is prior common knowledge among the players. However, each actual realized value t_i is private knowledge of the corresponding i-th player. There is also a set of possible *outcomes* O, over which the players have different preferences, represented by a valuation function which of course will depend also on her own realized type t_i, $v_i : T_i \times O \longrightarrow \mathbb{R}$. Think of $v_i(t_i, o)$ as the amount of "happiness" that agent i will receive if she happens to be of type t_i and outcome o occurs.

Each player i has also a set of available actions A_i. Let $A = \prod_{i=1}^{n} A_i$. A *mechanism* $\mathcal{M}$ comprises of a *decision* rule $x : A \longrightarrow O$ and a *payment* rule $p_i : A \longrightarrow \mathbb{R}$ for each player. The way to interpret this is that, after observing the joint action profile $\mathbf{a} = (a_1, \ldots, a_n)$ of the players, a mechanism selects an outcome $x(\mathbf{a}) \in O$ and charges each player i an amount of $p_i(\mathbf{a})$. Then, the personal gain of that agent is captured by her *utility function* $u_i : T_i \times A \longrightarrow \mathbb{R}$ defined by

$$u_i(t_i; \mathbf{a}) \equiv v_i(t_i, x(\mathbf{a})) - p_i(\mathbf{a}). \tag{2.1}$$

Each player thus, being fully rational and selfish, has to choose among her available actions A_i in order to maximize her own utility function. However, this has to be done in a Bayesian way of incomplete information: although the decision and allocation rules of the mechanism are common public knowledge, and the same holds for the valuation functions of the players, each player knows only her own type t_i while her beliefs about the others are limited to her prior knowledge of distribution F. The important observation here is that every mechanism induces an incomplete information game, in the sense of the previous Section 2.2.1, so the strategic behaviour of the players participating in the mechanisms, their actions and the resulting outcome, can be all analyzed in terms of that underlying game.

2.2.3 Truthfulness

Mechanisms can in general be rather involved, both computationally and structurally, since a great amount of flexibility is allowed by the above definition. Admittedly though, a most natural environment would be one at which the available actions to the players are just to report their type; in a single-item auction setting for example, like the one described in Section 1.1, we can think of the type t_i of the agent representing her personal value for the item and her layer of interaction with the selling mechanism being the submission of a bid b_i to the auctioneer. We call such mechanisms *direct-revelation mechanisms*, and they are formally defined by having the property that $A_i = T_i$.

Notice that in general these two values above, the actual type t_i and the reported one b_i, need *not* to be equal. In fact, since agents are rational and selfish, they can and will lie and misreport $b_i \neq t_i$ if this is to increase their own personal utility, given by (2.1). The advantages of a mechanism that would have the property of eliminating such behaviour by not incentivizing participants to lie are obvious: not only the protocol designer can then be sure that the received input consists of the actual private information of the agents, and thus implement his desired outcome in a straightforward way, but also the strategic layer of the game where the players are expected to reach an equilibrium through an abstract, possibly computationally infeasible process [23], is removed; *the participants know in advance that truth-telling is optimal for them and so they should not deviate from that*:

Definition 2.4 (Truthfulness)**.** A direct-revelation mechanism will be called *truthful* (or *incentive compatible (IC)*[1]*)* if truth-telling is an equilibrium of the induced game. Formally, the identity function over her type space is a dominant strategy for each player, i.e.

$$u_i\left(t_i;\left(t_i,\mathbf{t}_{-i}\right)\right) \geq u_i\left(t_i';\left(t_i,\mathbf{t}_{-i}\right)\right) \quad \text{for all } \; t_i, t_i' \in T_i \; \text{ and } \; \mathbf{t}_{-i} \in T_{-i}. \tag{2.2}$$

The notion of truthfulness can also be defined with respect to the alternative, weaker notion of Bayes-Nash equilibrium discussed in the previous Section 2.2.1, in which case it is called *Bayesian incentive compatibility (BIC)* and condition (2.2) is naturally adapted to:

$$\mathbb{E}_{\mathbf{t}_{-i}\sim F_{-i}|t_i}\left[u_i\left(t_i;\left(t_i,\mathbf{t}_{-i}\right)\right)\right] \geq \mathbb{E}_{\mathbf{t}_{-i}\sim F_{-i}|t_i}\left[u_i\left(t_i';\left(t_i,\mathbf{t}_{-i}\right)\right)\right] \quad \text{for all } \; t_i, t_i' \in T_i. \tag{2.3}$$

No matter how useful and desirable truthfulness might be, both socially and with regards to practical implementation considerations, it still raises a major concern: does this considerable restriction of the general class of non-direct and non-truthful mechanisms come also with a limitation on their power? Put another way, does this specialization restrict the ability of the mechanism designer with respect to what outcomes he can implement on the system? It turns out that this is not the case, due to the following celebrated result:

Theorem 2.1 (Revelation Principle [58])**.** *If a mechanism implements a particular outcome on equilibrium, then there is an equivalent direct-revelation, truthful mechanism*

[1] In the literature this is also denoted as DSIC (*dominant strategy* incentive compatibility), in order to emphasize the equilibrium notion under which it is achieved.

that implements the same outcome on (the truth-telling) equilibrium.

Formally, if for a mechanism $\mathcal{M} = (x, \mathbf{p})$ there is an *equilibrium* of strategies $\mathbf{s} = (s_1, \dots, s_n)$, then there is a *truthful* mechanism $\mathcal{M}' = (x', \mathbf{p}')$ such that

$$x'(\mathbf{t}) = x(\mathbf{s}(\mathbf{t})) \qquad \text{and} \qquad \mathbf{p}'(\mathbf{t}) = \mathbf{p}(\mathbf{s}(\mathbf{t})),$$

for all possible type profiles $\mathbf{t} \in T$, where $\mathbf{s}(\mathbf{t}) \equiv (s_1(t_1), \dots, s_n(t_n))$ and $\mathbf{p}(\mathbf{t}) \equiv (p_1(\mathbf{t}), \dots, p_n(\mathbf{t}))$. Observe that the two mechanisms $\mathcal{M}$ and $\mathcal{M}'$ not only select the same outcome but also use *the same payment rule*. This is very important for our analysis in the following chapters of the thesis, since we deal with revenue maximization. The essence of the revelation principle is that we can use truthful mechanisms to achieve the same goals, and in the same way, as with general mechanism. The Revelation Principle holds also in the Bayesian sense: we can use Bayes-Nash equilibrium implementation instead of dominant strategies and just replace DSIC with BIC truthfulness. The proof of Theorem 2.1 is so fundamental that may be considered trivial: essentially $\mathcal{M}'$ simulates the strategy equilibrium $\mathbf{s}$ of the players "internally" so that they don't need to strategize themselves, rendering misreporting meaningless.

In view of the Revelation Principle, from now on we will focus exclusively in the study of direct-revelation and truthful mechanisms, and study their properties on the truth-telling equilibrium. Another very natural property that we would like our mechanisms to satisfy is the following:

Definition 2.5. (Individual rationality) A truthful mechanism $\mathcal{M} = (x, \mathbf{p})$ will be called *individually rational (IR)* if it always induces nonnegative utilities for the players (at equilibrium). Formally, for every $i \in [n]$

$$u_i(t_i; \mathbf{t}) \geq 0 \qquad \text{for all } \ t_i \in T_i, \mathbf{t}_{-i} \in T_{-i}.$$

This property makes sure that no player can harm herself by honestly participating in the mechanism. Otherwise, she would be better of just "staying home". This is why IR is also known as *voluntary participation*.

2.2.4 Welfare Maximization and the VCG Mechanism

We mentioned in the introduction of Section 2.2.2 that the objective of Mechanism Design is to implement particular outcomes under the strategic interaction of the various players, who can have different and contradicting incentives. Perhaps the most natural such goal is maximizing the "collective happiness" of the society, which is represented by the sum of the player valuations $\sum_{i=1}^{n} v_i(t_i, o)$ for a given outcome $o \in O$ and agent types $\mathbf{t} \in T$. So, we define the *(social) welfare* of a Bayesian (direct-revelation, truthful) mechanism $\mathcal{M} = (x, \mathbf{p})$ as

$$\mathcal{W}(x) \equiv \mathop{\mathbb{E}}_{\mathbf{t}\sim F}\left[\sum_{i=1}^{n} v_i(t_i, x(\mathbf{t}))\right],$$

where F is the prior common knowledge distribution over the type space T. Mechanisms that maximize social welfare, that is

$$x(\mathbf{t}) \in \operatorname*{argmax}_{o\in\mathcal{O}} \sum_{i=1}^{n} v_i(t_i, o), \tag{2.4}$$

are called *efficient*. For instance, the second-price auction we discussed briefly in the introductory Chapter 1 is efficient, by definition: it allocates the item to the bidder with the highest value for it[2]. This Vickrey auction, which is an application of the second price design paradigm, is a special case of the general *VCG mechanism*: VCG is defined by the allocation rule given by (2.4), thus is efficient *by definition*, and deploys the following payment scheme that makes it truthful,

$$p_i(\mathbf{t}) = \max_{o\in O} \sum_{j\in[n]\setminus\{i\}} v_j(t_j, o) - \sum_{j\in[n]\setminus\{i\}} v_j(t_j, x(\mathbf{t})).$$

In economics terms, this payment rule can be understood as *internalizing the externalities* of every player, that is we ask each player to compensate us for the "harm" her presence in the mechanism setting causes to the rest of the society. The naming of the VCG mechanism is a tribute to the work of Vickrey [77], Clarke [20] and Groves [35].

Not all social objectives are (truthfully) implementable and, as a matter of fact, a celebrated result of Roberts [69] states that if we put no restrictions on the space of player valuations, and we have at least three available alternatives $|O| \geq 3$, then the *only* truthful mechanisms are *affine maximizers*, which are essentially just weighted variations of the VCG mechanism. Of course, this does not mean that no other social objectives can be achieved in special, restricted mechanism design settings: for example, as we will see in the following chapters, the space of truthful mechanisms is definitely richer in auction settings. Furthermore, there is a fine point worth mentioning here: Roberts' theorem (as well as our own treatment of Mechanism Design up to now in this chapter) applies to *deterministic* mechanisms; however, one can allow for probability distributions over such mechanisms and effectively get a model of *randomized* Mechanism Design that can potentially be much more powerful, something that we'll do in the remaining chapters.

For a more thorough treatment of the exciting area of Mechanism Design we recommend the texts of Fudenberg and Tirole [28, Chapters 6,7], Mas-Colell et al. [55, chapter 23], Narahari [60] and Vohra [78]. For an algorithmic flavour we refer to Nisan

[2]Notice that, although the first-price auction does the same, it is *not* truthful because of the different price rule.

[62], Hartline [42] and of course the seminal paper of Nisan and Ronen [63] that laid the foundations for the entire field now known as *Algorithmic Mechanism Design*.

2.3 Auction Theory

Now it's time for us to focus on the specific mechanism design environments that form the subject matter of this thesis, namely auctions. In an auction setting there are n players interested in buying m available goods. The set of available outcomes are the different allocations of items to buyers, that is[3] $O = [n+1]^{[m]}$, and the type t_i of each player i encodes the information about her preferences over these outcomes, parametrizing her valuation $v_i(t_i, S_i)$, where $S_i \subseteq [m]$ is the set of items allocated to her.

In general *combinatorial auctions*, no further restrictions are placed on these valuation functions. However, there are some natural special cases of particular interest, for example *additive valuations*, where the type t_i of a player is made up by m separate values $x_{i,j}$ for each item $j \in [m]$, representing bidder's i happiness in case of receiving good j, and the total valuation is just the sum of the values of the received items: $v_i(\mathbf{x}_i; S_i) = \sum_{j \in S_i} x_{i,j}$. Another important type of valuations is *unit-demand* ones, where each buyer is interested only in getting one of the items: $v_i(\mathbf{x}_i; S_i) = \max_{j \in S_i} x_{i,j}$. *In the present thesis we will be exclusively studying the first kind of additive auctions:* an intuitive way of thinking about them is that the goods for sale are *heterogeneous* and that they cannot substitute each other, so the acquisition of one does not affect the buyer's valuation for the others.

In light of the Revelation Principle (see Theorem 2.1), we will be (without loss) focusing on truthful (IC), direct-revelation selling mechanisms. We will also require Individual Rationality (IR, see Definition 2.5). So, to fix our notation, every buyer i will submit bids $x_{i,j}$ for all items $j \in [m]$, where $x_{i,j}$ belongs to some interval $D_{i,j} = [L_{i,j}, H_{i,j}] \subseteq \mathbb{R}_+$. Let $D_i = \prod_{j=1}^m D_{i,j}$ and $D = \prod_{i=1}^n D_i$. An auction is a protocol that, after receiving the players' bids $\mathbf{x}$, computes *allocation* and *payment* rules, $\mathbf{a} : D \longrightarrow I^{n \times m}$ and $\mathbf{p} : D \longrightarrow \mathbb{R}^n$, respectively; $a_{i,j}(\mathbf{x})$ is the probability with which item j is given to bidder i and $p_i(\mathbf{x})$ is the payment player i is asked to submit to the seller. In the case that one wants to consider only deterministic auctions, just take $a_{i,j}(\mathbf{x}) \in \{0, 1\}$. In order for the allocation probabilities to be well defined, we need to make sure that for all items $j \in [m]$:

$$\sum_{i=1}^n a_{i,j}(\mathbf{x}) \leq 1 \qquad \text{for all } \mathbf{x} \in D.$$

Then, because of additive valuations, the *utility* function of each player (see (2.1)) is

[3] The seller is not obligated to allocate all items, and that's where $[n+1]$ comes from.

given by

$$u_i(\mathbf{x}) = \mathbf{a}_i(\mathbf{x}) \cdot \mathbf{x}_i - p_i(\mathbf{x}) = \sum_{j=1}^{m} a_{i,j}(\mathbf{x}) x_{i,j} - p_i(\mathbf{x}). \tag{2.5}$$

This captures the expected (under the randomization of the auction) gain of player i: the expected sum of the values of the item she manages to purchase, minus the payment she has to submit to the auctioneer[4]. Notice that the IR constraint requires these utilities to be nonnegative. On the other hand, the seller's own gain is captured by the total *revenue* of the auction

$$\sum_{i=1}^{n} p_i(\mathbf{x}) = \sum_{i=1}^{n} \left(\mathbf{a}_i(\mathbf{x}) \cdot \mathbf{x}_i - u_i(\mathbf{x})\right), \tag{2.6}$$

which is a simple rearrangement of (2.5).

The following is an elegant, extremely useful analytic characterization of truthful mechanisms due to Rochet [70]. For a proof of this we recommend [38].

Theorem 2.2. *An auction $\mathcal{M} = (\mathbf{a}, \mathbf{p})$ is truthful (IC) if and only if the utility functions u_i that $\mathcal{M}$ induces have the following properties with respect to the i-th row components, for all bidders i:*

1. *$u_i(\mathbf{x}_{-i}, \cdot)$ is a* convex *function*
2. *$u_i(\mathbf{x}_{-i}, \cdot)$ is almost everywhere differentiable with*

$$\frac{\partial u_i(\mathbf{x})}{\partial x_{i,j}} = a_{i,j}(\mathbf{x}) \quad \text{for all items } j \in [m] \text{ and a.e. } \mathbf{x} \in D. \tag{2.7}$$

The allocation function $\mathbf{a}_i$ is a subgradient[5] *of u_i.*

Theorem 2.2 essentially establishes a kind of correspondence between truthful mechanisms and utility functions. Not only every auction induces well-defined utility functions for the bidders, but also conversely, given nonnegative convex functions that satisfy the properties of the theorem we can fully recover a corresponding mechanism from expressions (2.7) and (2.6).

2.3.1 Revenue Maximization

There is a Bayesian layer of incomplete information in our environment, given by a prior joint distribution F over the bid space D, according to which $\mathbf{x}$ is going to be realized. Recall that, although F is common knowledge among the players and the auctioneer, the actual realization of the bid vectors $\mathbf{x}_i$ are only privately observed by the respective player i.

[4]We will only consider *risk neutral* players.

[5]For reference to Convex Analysis notions we recommend the classic text of Rockafellar [72].

In this thesis we study the problem of maximizing the seller's expected revenue based on his prior knowledge of F, under the IR and IC constraints, thus (by Theorem 2.2 and (2.6)) maximizing

$$\mathcal{R}(\mathbf{u};F) \equiv \sum_{i\in[n]} \int_D \left(\nabla u_i(\mathbf{x})\cdot \mathbf{x}_i - u_i(\mathbf{x})\right)\, dF(\mathbf{x}) \tag{2.8}$$

over the space of *nonnegative convex functions* u_i on D having the properties

$$\sum_{i\in[n]} \frac{\partial u_i(\mathbf{x})}{\partial x_{i,j}} \leq 1 \tag{2.9}$$

$$\frac{\partial u_i(\mathbf{x})}{\partial x_{i,j}} \geq 0 \tag{2.10}$$

for a.e. $\mathbf{x}\in D$, all $i\in[n]$ and $j\in[m]$. The *optimal* revenue is defined as

$$\textsc{Rev}(F) \equiv \sup_{\mathbf{u}} \mathcal{R}(\mathbf{u};F),$$

that is the maximum revenue among all feasible truthful mechanisms. A mechanism inducing utilities $\mathbf{u}$ for which $\mathcal{R}(\mathbf{u};F)=\textsc{Rev}(F)$ will be called an *optimal auction*.

2.3.2 Myerson's Optimal Auction

In this section we briefly overview some of the notions and results from the seminal work of Myerson [58] that will be necessary for our exposition. Recall that the holy grail of multidimensional optimal auction design is exactly to generalize, in the "right" way, these elegant ideas to multiple-good settings.

Definition 2.6 (Regular distributions)**.** A probability distribution F over a real domain will be called *regular* if the *virtual value* function, defined by

$$\phi(x) \equiv x - \frac{1-F(x)}{f(x)},$$

is (weakly) increasing, where f is the distribution's density.

Definition 2.7 (Monotone hazard rate)**.** A probability distribution F over a real domain is said to have *monotone hazard rate (MHR)* if quantity

$$\frac{f(x)}{1-F(x)}$$

is increasing.

It is trivial to see that every MHR distribution is also regular. The inverse however is not true, as demonstrated by the following important probability distribution:

Definition 2.8 (Equal-revenue distribution)**.** The probability distribution over $[1, \infty)$ with density and cumulative functions

$$f(x) = \frac{1}{x^2} \qquad \text{and} \qquad F(x) = 1 - \frac{1}{x},$$

respectively, is called *equal revenue (ER)*.

The hazard rate of the ER is $\frac{1/x^2}{1/x} = \frac{1}{x}$ which is decreasing (and thus it is not MHR) and its virtual valuation is constant $\phi(x) = x - x = 0$, placing it in the "boundary" of regularity. Also, it is important to mention here that ER has unbounded expectation: $\int_1^\infty x f(x)\, dx = \int_1^\infty \frac{1}{x}\, dx = \infty$. These properties make ER a typical counter-example in disproving general auction properties, or a test-bed for developing intuition and testing the validity of others. The reason behind the particular naming will become clear very soon.

Theorem 2.3 (Myerson [58])**.** *In a single-good setting where the values of the buyers for the item follow independent regular distributions, the optimal auction allocates the good by performing a Vickrey auction on the* virtual *values (see Definition 2.7).*

To understand more clearly how Myerson's auction performs, consider the case of n identical bidders whose values for the items come from the same distribution F. Then, a consequence of Theorem 2.3 is that the optimal auction is a second-price one with an added *reserve price* equal to the root[6] of the virtual value function $\phi(x)$. This means, that the seller essentially inserts into the auction a new "dummy" player with a value of $r = \phi^{-1}(0)$. If she is the winning bidder, then the good remains unsold. Think for example the typical ascending price auctions where the auction-house sets a starting price r and the item is sold to the last remaining bidder.

The importance of Myerson's result lies not only in providing a simple, elegant and closed-form description of the optimal auction, but also in the fact that this auction turns out to be *deterministic*. This is something not trivial at all, since our initial revenue-maximization is taken with respect to all feasible truthful auctions, including *lotteries*. Consider the simple scenario of a seller with a single good facing a buyer, whose value for the item comes from a regular distribution F. Take this distribution to be the uniform over the unit interval I. Then, the virtual value is $\phi(x) = x - \frac{1-x}{1} = 2x - 1$, so the optimal selling mechanism is for the auctioneer to just offer a *take-it-or-leave-it price* of $r = \frac{1}{2}$, the root of $\phi(x)$. This results to an (optimal) revenue of $\text{MRev}(\mathcal{U}) = \frac{1}{4}$, since the bid has $\frac{1}{2}$ probability of being above r. Notice that here we have used the notation MRev of Hart and Nisan [40] instead of the standard Rev. We will do that sometimes whenever we want to emphasize the fact that this optimization is within a single-item Myersonian auction setting. For the exponential distribution

[6] If $\phi(x)$ is not *strictly* increasing it might be the case that it has multiple roots (see for example the equal revenue distribution in Definition 2.8). In such a case the roots form a real interval, and we just take the left boundary of it, i.e. the smallest root.

$\mathcal{E}(\lambda)$ we would have $\phi(x) = x - \frac{1-\left(1-e^{-\lambda x}\right)}{\lambda e^{-\lambda x}} = x - \frac{1}{\lambda}$, resulting to a price of $r = \frac{1}{\lambda}$ for a revenue of $\frac{1}{\lambda e}$. In general, for this single-bidder case, the optimal revenue is thus simply given by

$$\text{MRev}(F) = \max_r r(1 - F(r)).$$

For the equal revenue distribution in particular, every choice of $r \in [1, \infty)$ will result in the same revenue for the seller, namely $\text{MRev}(\text{ER}) = r\left(1 - \left(1 - \frac{1}{r}\right)\right) = 1$, which also explains the naming. Formally, Myerson's auction in this setting will choose the smallest price, that is $r = 1$.

The regularity assumption in Theorem 2.3 is not critical: the theorem can be appropriately modified to deal with non-regular priors, through an *ironing* process of the virtual values proposed by Myerson [58]. However the same is not true for the independence requirement which is essential, and a great deal of research effort has been devoted into designing auctions that can handle correlation among bidders, mostly by approximating the optimal revenue through welfare, using simple, Vickrey-based auctions (see e.g. [42, Chapter 4]).

One important, general remark about auction design we would like to make here regards *tie-breaking*. For example, what happens in Myerson's auctions if two bidders have exactly the same nonnegative virtual values, which are both maximum? Who is going to win the item? In such a scenario, we can e.g. break ties uniformly at random without affecting the auction's revenue or truthfulness. However, it is crucial to remember here that if the probability distributions have well-defined density functions (which, by the way, is an a priori requirement for regularity due to Definition 2.6) then "problematic" events like the previous have *zero* probability measure, thus they are essentially insignificant for our analysis.

2.3.3 Some Easy Approximation Results

In this section we return to a general setting of n bidders and m goods, with values coming from independent probability distributions $F_{i,j}$. Item-specific priors are denoted by $F_j = \prod_{i=1}^{n} F_{i,j}$, and the full joint distribution is denoted by $F = \prod_{j=1}^{m} F_j$.

In the previous section we talked about Myerson's elegant solution for maximizing the revenue in single-item auctions (with independent bidders). So, an obvious thing to wonder is whether these results can be generalized in a simple, direct way to multidimensional settings involving many goods. As it will become apparent soon enough, and after reviewing the related work in Section 2.4, this problem turns out to be extremely challenging and, as a matter of fact, forms the very subject of the present thesis.

But as a first attempt, it seems natural trying to reduce multi- to single-item auctions in a direct way, so that one can use Myerson's formulas. After all, if we know how to sell *optimally* each item independently from the others, and the values for the items come from independent distributions, then it makes sense to hope that selling each item *separately* may be indeed optimal. Unfortunately that is not the case as we will see shortly, but still it gives one of the simplest and most important multiple-item auctions, which is also deterministic. We denote its revenue by SRev, i.e.

$$\mathrm{SRev}(F) \equiv \mathrm{MRev}(F_1) + \mathrm{MRev}(F_2) + \cdots + \mathrm{MRev}(F_m). \tag{2.11}$$

At the other end of the spectrum, lies another simple deterministic auction, putting all items together and offering them as a single *full bundle*:

$$\mathrm{BRev}(F) \equiv \mathrm{MRev}(F_1 * F_2 * \cdots * F_m).$$

Here $F_1 * F_2 * \cdots * F_m \equiv F_S$ is the *convolution* of distributions $F_1, F_2, \ldots, F_m$, i.e. if $X_j \sim F_j$ are the n-dimensional random variables representing the item values, then F_S is the distribution of the sum of the valuations: $S = \sum_{j=1}^{m} X_j$.

In particular, following the discussion in the previous Section 2.3.2 about single-buyer settings, if $F_j \sim \mathcal{U}$ for all $j \in [m]$, we have

$$\mathrm{SRev}(\mathcal{U}^m) = \frac{m}{4} \quad \text{and} \quad \mathrm{BRev}(\mathcal{U}^m) = \sup_{x \in [0,m]} x(1 - F_S(x)), \tag{2.12}$$

where F_S is the Irwin-Hall distribution of the sum of m independent uniform random variables over I, i.e. (see [37])

$$F_S(x) = \frac{1}{m!} \sum_{k=0}^{\lfloor x \rfloor} (-1)^k \binom{m}{k} (x-k)^m, \quad 0 \le x \le m. \tag{2.13}$$

In the same way, for independent exponential and for i.i.d. exponential goods, we can see that

$$\mathrm{SRev}(\mathcal{E}) = \frac{1}{e} \sum_{j=1}^{m} \frac{1}{\lambda_j} \quad \text{and} \quad \mathrm{SRev}(\mathcal{E}^m(\lambda)) = \frac{m}{\lambda e}. \tag{2.14}$$

These two important deterministic selling mechanisms were extensively studied by Hart and Nisan [38], who essentially sparked again the interest of the theoretical computer science community for optimal multidimensional auctions.

Selling the items separately may not be, in general, the optimal selling mechanism when more than a single item is to be sold. Take for example the case of a single bidder and two uniformly i.i.d. items over $[0,1]$. The optimal way to sell them separately is by setting a price of $\frac{1}{2}$ for each one, resulting in an expetected revenue of $\frac{2}{4} = \frac{1}{2}$ (from (2.12)). But if we offer the items in a bundle for a price of p, with $0 \le p \le 1$, then the revenue becomes $p\left(1 - \frac{p^2}{2}\right)$; the probability of the sum of two i.i.d. uniform random variables being less than p equals $\frac{p^2}{2}$. By selecting $p = \frac{3}{4}$ this gives an expected revenue of $\frac{69}{128}$ which is strictly better than $\frac{1}{2}$. Intuitively, this demonstrates how a seller may have an incentive to offer bundles of items at a *discounted* price ($p = \frac{3}{4} <$

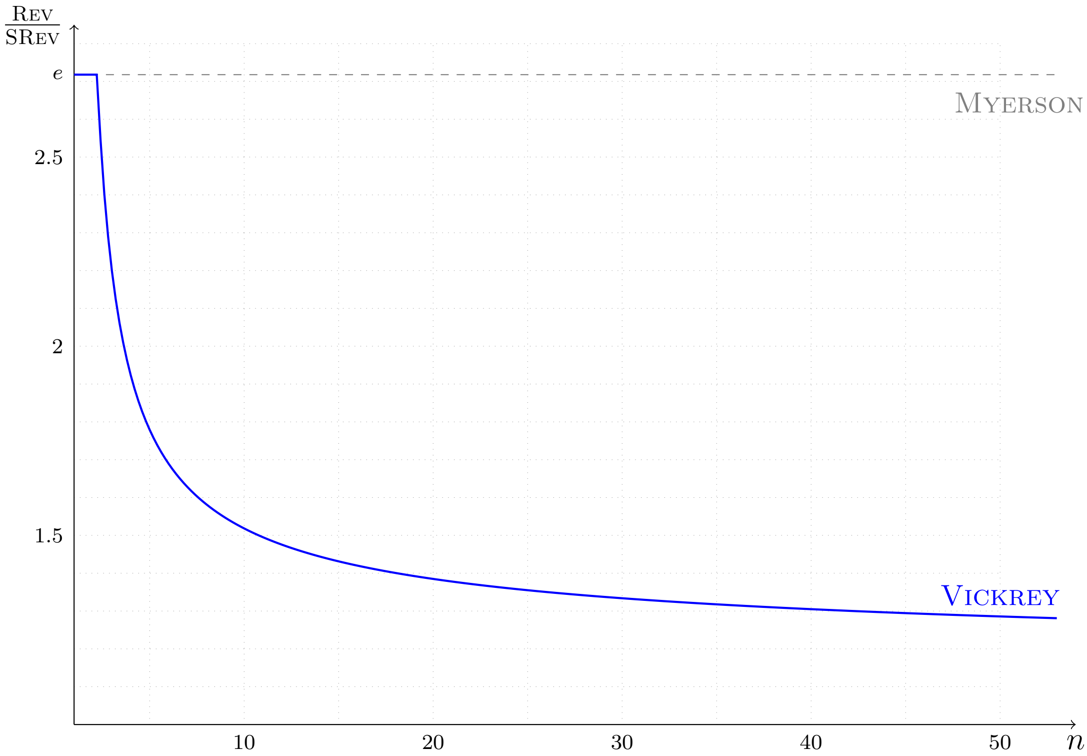

Figure 2.1: The upper bound on the approximation ratio of selling separately when the values come from independent MHR distributions, as described in Theorem 2.4, drawn for $n = 1, 2, \dots, 50$ players (and any number of items m). In general, an e-approximation (grey line) can be guaranteed by using Myerson's optimal auction. In the special case of identical bidders, the analysis can be improved further by considering a simple Vickrey auction (blue line).

$\frac{1}{2} + \frac{1}{2}$) instead of selling them separately. However, as we show next, under some regularity assumptions, it can still provide good, constant approximation ratios of the optimal revenue. Furthermore, these guarantees can be achieved in almost trivial ways, by using as subroutines known welfare-approximation techniques from single-good environments:

Theorem 2.4. *If the values of the players for the items come from MHR distributions, then*

$$\textsc{SRev}(F) \geq \frac{1}{e} \cdot \textsc{Rev}(F).$$

Furthermore, if bidders are identical[7] *then using the Vickrey auction to sell separately every good is* $1 + \frac{1}{H_n - 1} = 1 + O(\ln^{-1} n)$*–approximate to the optimal revenue. A plot of this approximation ratio can be seen in Figure 2.1.*

In order to prove the theorem we will need the following properties of MHR distributions, the proofs of which can be found in [47, Appendix C] and [5, Lemma 13]:

Lemma 2.1. *For any nonnegative MHR-distributed random variable* X*,*

$$\mathbb{E}[X^{(n-1:n)}] = n\,\mathbb{E}[X^{(n-1:n-1)}] - (n-1)\,\mathbb{E}[X^{(n:n)}] \tag{2.15}$$

[7] That is, $F_{i,j} = F_{i',j}$ for all players $i, i' \in [n]$ and goods $j \in [m]$.

and

$$\mathbb{E}[X^{(n-1:n-1)}] \geq \frac{H_{n-1}}{H_n}\,\mathbb{E}[X^{(n:n)}]. \tag{2.16}$$

Proof of Theorem 2.4. First consider the case of identical bidders and let $X_j \sim F_j$ be the random variable of the value of good j. From the IR constraint, the optimal revenue cannot exceed the maximum social welfare,

$$\textsc{Rev}(F) \leq \sum_{j=1}^{m} \mathbb{E}\left[{X_j}^{(n:n)}\right]$$

and by selling all items separately using the Vickrey auction gives

$$\textsc{SRev}(F) \geq \sum_{j=1}^{m} \mathbb{E}[{X_j}^{(n-1:n)}],$$

since each good is allocated to the second-best bidder. Thus, by using (2.15) and (2.16) we get an approximation ratio of

$$\begin{aligned}
\frac{\textsc{Rev}(F)}{\textsc{SRev}(F)} &\leq \max_{j\in[m]} \frac{\mathbb{E}\left[{X_j}^{(n:n)}\right]}{\mathbb{E}[{X_j}^{(n-1:n)}]} \\
&\leq \max_{j\in[m]} \frac{\mathbb{E}\left[{X_j}^{(n:n)}\right]}{n\,\mathbb{E}[{X_j}^{(n-1:n-1)}] - (n-1)\,\mathbb{E}[{X_j}^{(n:n)}]} \\
&\leq \max_{j\in[m]} \frac{\mathbb{E}\left[{X_j}^{(n:n)}\right]}{n\frac{H_{n-1}}{H_n}\,\mathbb{E}[{X_j}^{(n:n)}] - (n-1)\,\mathbb{E}[{X_j}^{(n:n)}]} \\
&= \frac{1}{n\frac{H_{n-1}}{H_n} - n + 1} \\
&= 1 + \frac{1}{H_n - 1}
\end{aligned}$$

In case of non-identical bidders, instead of the Vickrey auction we can use any other single-item auction whose revenue can approximate well the maximum welfare at every step; this can be achieved for example by Myerson's optimal auction (with reserve prices) for a factor of e (see [27, Theorem 3.11] for a proof of that). □

To the best of our knowledge the statement of Theorem 2.4 as well as its proof appear here for the first time. However, it may be considered folklore among some researchers in the field of Optimal Auction Design.

2.4 Related Work

In this section we will briefly survey some of the previous work related to the subject of the thesis. Our choice is more in favour of depth, rather than breadth: the literature in optimal auction design has seen an explosion in the last years, and on top of that it often comes from different and sometimes disconnected research communities, resulting in serious difficulties for a third-party to get quickly a clear understanding of the current state and the various underlying details of open problems. This is exactly our goal here: to follow this virtual thread, the storyline leading from Myerson's seminal work to present-day active research, keeping at all times our point of view set upon the subject of *multidimensional* revenue maximization (primarily under additive valuations), with intermediate stops at what we consider to be the most relevant and important results. This should prove most useful, for example, to a researcher interested in working in the field and who wants to get a fast and reliable introduction to the subject. As one would expect, our presentation is biased towards a theoretical computer science perspective. For more thorough and holistic treatments we recommend the excellent lecture notes of Hartline [42], the relevant chapter from the AGT book [43] or the surveys of Klemperer [46, 45].

The solid foundations of the entire field of rigorous optimal auction design were set by the seminal work of Myerson [58] that we briefly talked about in Section 2.3.2; Myerson also later won the 2007 Nobel prize in Economics, partly for these contributions. Of course, some important results have been also developed earlier, for example by Vickrey [77], Clarke [20] and Groves [35] as we saw in Section 2.2. The elegant characterization of truthfulness via the sub-gradients of the utility functions (Theorem 2.2) is due to Rochet [70, 71]. One of the first solid attempts by economists to generalize Myerson's results to multiple goods when valuations are additive was by McAfee and McMillan [56]. They provided the following condition (see Equation (46) in [56]) on the density f of the buyer's prior joint distribution over the m-item values (which in general don't need to be independent)

$$(m+1)f(\mathbf{x}) + \mathbf{x} \cdot \nabla f(\mathbf{x}) \geq 0, \tag{2.17}$$

and proved that this condition is sufficient for deterministic pricing rules to be optimal (among all feasible auctions, including lotteries). However, this was later shown to be erroneous by [75] and [53], independently. Notice that for the special case of a single good this has a strong connection with the standard singe-dimensional regularity condition of Definition 2.6, since for $m = 1$ we get that $f(x)\left(x - \frac{1-F(x)}{f(x)}\right)$ is increasing, thus ensuring the single-crossing property of the virtual valuation function (see also the discussion in [53, Sect. 2]).

In particular, this initiated a long stream of results trying to demonstrate the limitations of determinism in general, by providing counterexamples where lotteries are necessary for optimality, under various single-buyer settings: Thanassoulis [75] (unit-demand valuations, correlated distributions), Manelli and Vincent [53] (two correlated items, supported over I^2), Pycia [67] (two correlated items with a finite support of size 2), Pavlov [66, Example 3(ii)] (two uniformly i.i.d. items over $[c, c+1]$ where

$c \in (0, 0.077)$), Hart and Reny [41] (two correlated items with a support of size 4; two i.i.d. items with a support of size 6) and Daskalakis et al. [25] (two non-identical but independent exponentially distributed items over $[0, \infty)$). The exposition of [41] is particularly straightforward and provides a clear intuition on the truthfulness aspects of the setting, something which is critical in auction design. Daskalakis et al. [25] present also another counterexample, involving two specific, non-identical beta distributions over $[0, 1)$; it is of great interest, since it turns out that the optimal auction has to use a continuum of randomized options, that is, the seller has to offer to the buyer *infinitely* many lotteries.

This is related to the notion of *menu-size complexity*, introduced earlier by Hart and Nisan [39] in order to quantify the degree of an auction's simplicity in terms of how many different alternatives are offered to the buyer. This structural approach allowed them to prove some powerful results in a very clear way: they give (correlated) distribution examples where the optimal revenue increases arbitrarily with the size of the menu and, in particular, deduce that by bounding the size of the menu no constant fraction of the optimal revenue can be guaranteed. Also, since deterministic auctions can have at most exponential (with respect to the number of items m) menu sizes, this is another demonstration of their sub-optimality. If we restrict our attention to bounded domains of the form $[1, H]^m$ though, they can show that just polylogarithmic (for $m = 2$) or polynomial (for $m > 2$) size menus are enough to achieve arbitrary approximations to the optimal revenue. This last observation links their results to the PTAS[8] of Daskalakis and Weinberg [22] where the key element of their construction is essentially the proof that small (polynomial) menu sizes are enough to provide good approximations. Hart and Nisan [39] also answer an open question from the standard-reference paper of Briest et al. [8] about unit-demand settings, showing that for the case of two items the gap between the performance of deterministic and optimal auctions is bounded[9], achieved through proving that the two models (that is, unit-demand and additive valuation) can differ at most exponentially.

Manelli and Vincent [53] were the first to give an *exactly* optimal solution in a multidimensional (single-buyer) additive setting, about 25 years after Myerson's paper. They provided sufficient conditions for optimality of deterministic auctions and also some necessary ones, but only within the class of price schedules, and not in the general class of IC and IR auctions (which includes lotteries). From this, for example, they can show that the optimal deterministic auction for m uniformly distributed goods must set a price of $\frac{m}{m+1}$ for the 1-item bundles. All these conditions are topological in nature, in the form of functional inequalities over abstract partitions of the allocation space, and admittedly difficult to interpret. However, they were able to instantiate them for

[8] *Polynomial Time Approximation Scheme* [76]: it means that for every $\varepsilon > 0$ there exists an algorithm that computes a solution which is within a (multiplicative) factor of $(1 + \varepsilon)$ from the optimal one, having polynomial running time (where ε is considered a constant).

[9] It was already known by [8] that this is not the case for unit-demand settings with more items.

two and three items into specific requirements for the probability prior and, although still involved, they showed that they are satisfied by the uniform distribution, thus establishing exact optimality.

Their methodology is inspired by some earlier work of theirs [52] and one can also see some underlying hints to duality, although not in the straightforward way of later works [25, 31]. Throughout their exposition, Manelli and Vincent [53] assume that the joint distribution has a continuously differentiable density f in the unit hypercube I^m and that $f(\mathbf{x}) + x_j \frac{\partial f(\mathbf{x})}{\partial x_j} \geq 0$ for all items j, which can be translated essentially to $x_i f(\mathbf{x})$ being (weakly) increasing with respect to all coordinates. Since for the most part they deal with independent valuations, f is a product distribution and so the above conditions are translated to each f_j being continuously differentiable in I and $x f_j(x)$ being increasing. This is stronger than the McAfee and McMillan [56] assumption (2.17), which in turn, for just $m = 1$ item is the regularity condition of Definition 2.6. When they instantiate their method for two items they require the extra condition that $\frac{x f_j'(x)}{f_j(x)}$ is increasing, which they admit it is unusual and seems to arise due to the "multidimensional character of the problem". Their analysis for three items is done only for the uniform distribution specifically, because the optimality conditions become even more difficult to manipulate and their verification is done via direct computation methods for finding the best price schedule and then showing that these prices also satisfy the sufficient conditions.

In follow-up work, Manelli and Vincent [54] perform a complete topological analysis of the revenue optimization problem. They approach it as maximizing a linear functional over a (convex) functional space of feasible utility functions. By Bauer's Maximum Principle (see e.g. [2, Chapter 7]) this means that an optimal mechanism is an extreme point of this space. However, it is not true that every extreme point is a maximizer and in fact this is the very difficulty of the problem itself: the richness of these points and the challenge to discriminate maximizers from general extreme points. They describe the extreme points of the feasibility space giving algebraic conditions and also give sufficient conditions for a mechanism to be an extreme point. However, minimizers are also extreme points. Up to this stage their analysis does not take into consideration the valuation priors at all, so it is valid for arbitrary joint distributions. Next they assume independent items, introduce a notion of domination (with respect to the induced revenue) and show that every undominated extreme point mechanism is optimal for some product distribution (however, as Hart and Reny [41] point out, there is an error in the proof, in particular the assumption that the set of product distributions (named G in their proof of Theorem 9) is convex).

Li and Yao [48] show that selling m goods separately is $O(\log m)$–approximate and furthermore, if the items have identical distributions, full-bundling has constant approximation ratio. This is an improvement upon the work of Hart and Nisan [38] that gave $O(\log^2 m)$ and $U(\log m)$ bounds, respectively. Li and Yao [48] also show that their bounds are tight up to constant factors. Their results require *no* regularity assumptions over the distributions, just independence, which is also the case in the work of Hart and Nisan [38].

For the special case of two independent items Hart and Nisan [38] have proved an approximation ratio of 2 for selling separately, which goes down to $(1+e)/e \approx 1.368$ when the items are identical. They don't have a matching lower bound for the latter, and the best separation they present is of a factor of 1.278. In fact this is taken with respect to the full-bundle auction, since for the particular example they use (two equal-revenue i.i.d. goods) it can be shown that bundling is optimal. They also give a general sufficient condition for full-bundling optimality for two-item i.i.d. settings, namely

$$f'(x) + \frac{3}{2}f(x) \leq 0$$

for all values $x \geq \alpha$ after some positive point α, and $f(x) = 0$ before that. This condition is satisfied by the ER distribution. However, a calculation shows that power-law distributions $f(x) = \frac{1-c}{x^c}$, $x \in [1, \infty)$ where $c > 2$, which are also regular but *not* MHR, fail to satisfy this condition. They also show that if distributions are not independent then full-bundling's approximation ratio goes up to linear m and this is tight. They can, though, extend the 2-approximation result for two independent items to the case of n bidders. Finally, they provide an extensive discussion about the fact that by the law of large numbers, if we let $m \to \infty$ in the i.i.d. setting, then the revenue of the full-bundle auction approaches the optimal one (that is, m times the expectation of the distribution). However, as they also argue, this gives no *specific* approximation guarantees for fixed values of m. In particular, they can construct examples, no matter how large m is, where full-bundling is less than 0.57 of the revenue of the auction that sells items separately.

Pavlov [66] focuses his study in two-good environments for which the joint distribution also has to satisfy the regularity condition (2.17). He studies both the unit-demand and additive settings and gives some properties that the optimal auction must satisfy, which are of a structural nature, saying e.g. that at least one of the goods has to be sold deterministically with probability 1 (unless none of the items is sold). He applies this for uniform distributions over unit-length intervals of the form $[c, c+1]$, $c \geq 0$, and after properly manipulating these characterizations in an algebraic way, he shows that the optimal auction for two-items is deterministic, except for the case where $c \in (0, 0.077)$, and gives a complete, closed-form description of it.

Hart and Reny [41] show how optimal revenue in multi-good settings is *not* monotonic with respect to the player's valuations. Another very elegant contribution is that they propose a framework to study optimal auctions that avoids the technical complications of the fact that, in general, utility functions need not to be everywhere differentiable (but only *almost* everywhere) since they are convex. They do that by focusing on *seller-favourable* mechanisms which at every point of non-differentiability just pick the value of the subgradient (i.e. allocation, see Theorem 2.2) that favours the seller (that is, has the highest price). This is done by deploying directional derivatives (that exist everywhere), and taking the best direction for the seller. They show that this is a well-defined process which is without loss for the revenue-maximization objective.

In very recent, exciting work, Babaioff et al. [6] show that for arbitrary independent distributions, the best of the two among selling separately and selling in a full bundle, is always a constant approximation to the optimal, namely 7.5 (this was improved to 6 in a later version of the paper). Their approach is based on the core-tail separation technique introduced by Li and Yao [48]. They also extend their technique to *multiple-buyer* settings where they get a $O(\log m)$–approximation for the auction that sells separately. This is asymptotically tight (due to a lower bound by Hart and Nisan [38]). In fact, this was the first non-trivial positive result with an approximation ratio for general *multi-bidder* settings. They introduce the notion of a *partition mechanism* which first groups the items in different bundles and then sells these bundles separately, as single items. This is a generalization of both selling separately and in a full bundle. They show that when many bidders are involved, the optimal partition mechanism may be $\Omega(\log m)$ away from the optimal, but also $\Omega(\log m)$ better than the best of selling independently and in a full bundle. Finally, with respect to our standard single-buyer setting but under *correlated* items, they show that selling separately is $\Theta(\log m)$-approximate to the best partition mechanism. In fact, there is a $\Omega(\log m)$ separation between the optimal among selling separately and in a full bundle, and the best partition mechanism.

Menicucci et al. [57] show that for two i.i.d. goods over an interval $[a, b]$, if the prior distribution satisfies the regularity condition (2.17) of [56, 66], and also exists a constant $c > 1$ such that

$$xf(x) \geq c \quad \text{and} \quad F\left(\frac{a+b}{2}\right) \leq c - 1,$$

then full-bundling is optimal. A more simple, sufficient condition is for the density function f to be increasing and satisfy $af(a) \geq 3/2$. For example, the uniform distribution over intervals $[a, a+1]$ with $a \geq 3/2$ satisfies this condition. However, this is already known by the results of Pavlov [66]. They also provide weaker conditions under which full-bundling is optimal among *deterministic* price schedules.

Multiple buyers For multiple bidders and independent goods Yao [80] approaches the problem via reducing it to the single-item case, introducing what he calls *Best-Guess* reduction. He makes no regularity assumptions on the distributions, and shows the existence of a constant-approximate deterministic auction, generalizing thus the results of Babaioff et al. [6]. If, furthermore, the players are independent, then the optimal revenue achieved in DSIC is within a constant factor of the BIC one. Finally, if all item values are also identically distributed, then he expresses the optimal revenue in an asymptotic closed form, and describes an auction called *2nd-Price Bundling* that achieves this.

Duality A similar, in principle, approach to ours, proposing in a clear way a duality framework for multidimensional revenue maximization has been independently developed by Daskalakis et al. [25], and so a detailed coverage of their work is in order. They study the problem in the case of a single buyer and independent goods. They formulate a measure-theoretic complementarity-like sufficient condition for optimality. Since this condition is rather abstract and difficult to satisfy, they provide a tool which involves the stochastic dominance of certain measures and uses Strassen's theorem [49], and which is capable of showing the existence of a critical measure γ^* that is sufficient for complementarity. Using this, they were also able to provide a measure-theoretic condition that is sufficient for the optimality of the deterministic full-bundling auction. By focusing in the special case of only two items, they manage to manipulate these conditions and transform them into more simple, analytic in nature characterizations.

By deploying these tools they achieved to explicitly describe the optimal auction for selling two (not necessarily identical) exponentially distributed items, which turns out to be either deterministic or randomized, depending on the ratio between the parameters of the exponential distributions (see Appendix A.4 for our analysis of this auction). They also give closed-forms of the optimal auctions for two particular examples: one for two goods following power-law distributions with parameters $c_1 = 6$ and $c_2 = 7$, for which they show that full-bundling is optimal and numerically compute the optimal bundle price $p^* \approx 0.357$; and another for two goods from beta distributions $f_1(x) = \frac{x^2(1-x)^2}{B(3,3)}$ and $f_2(x) = \frac{x^2(1-x)^2}{B(3,4)}$, $x \in [0, 1)$, where the optimal auction is randomized. Notice that these two particular beta distributions are both regular, but f_2 is *not* MHR (see Definition 2.7). The exponential distributions, as we saw in Section 2.3.2, are MHR (thus also regular). The power-law distributions are also regular, but not MHR (in fact their monotone hazard rate is *strictly decreasing*).

Finally, an overview of the assumptions that Daskalakis et al. [25] impose on the probability distributions for the values of the goods: they are defined in semi-open intervals $[d^-, d^+)$, where $d^- \geq 0$ and d^+ possibly infinite, and they need to have a continuously differentiable density f that decreases more steeply than $1/x^2$, or more generally in the case of bounded supports, $\lim_{x \to d^+} x^2 f(x) = 0$; also f has to vanish on the left point of the interval, that is $d^- f(d^-) = 0$.

Algorithmic approximations By this term we mean results where efficient algorithms for finding an approximately optimal mechanism are given, instead of results proving good approximation ratios for particular (simple, with closed form descriptions) mechanisms (like e.g. in the work of Hart and Nisan [38] mentioned above). The two approaches may seem equivalent from an algorithmic/procedural perspective, however there is an essential difference: the first one does not necessarily provide a solid description of the particular mechanism that achieves the approximately optimal performance. On this, see also our discussion on *conceptual complexity* [40] in Item 7 of Chapter 7.

Daskalakis and Weinberg [22, 21] give a PTAS (over $\max\{n, m\}$) that locates a mechanism which is within an additive ϵ, in the case of bounded distributions, or multiplicative $(1 - \epsilon)$ factor, for the case of unbounded MHR ones, from the optimal revenue. They assume either a constant number of bidders that each one of them has identical values for the items (the distributions need not to be identical across bidders, but just independent), called the *item-symmetric* setting, or the number of items fixed and identical distributions across the bidders, which however can have arbitrary correlation among items, called the *bidder-symmetric* setting. They work in the unit interval and claim that this is w.l.o.g. for any general bounded set. One key feature of their work is that, if the distribution support is bounded, or unbounded but satisfies the MHR condition, then they provide a technique that can truncate and discretize these supports while still achieving nearly optimal ϵ-BIC mechanisms. This, in turn, can then be reduced to a nearly optimal BIC mechanism. This approach plays also a critical role in some other approximation results of Cai et al. [11] and Cai et al. [13] that study the *explicit setting*[10] and that have polynomial running-times on the size of the support in general: they allow them to run efficiently also for continuous supports, in the special case of item-symmetric distributions (see e.g. [11, Section 3.3]).

Cai and Daskalakis [9] essentially study the same approach for unit-demand, single-bidder settings; a combined review of these two works can be found in [12]. Cai and Huang [10], like Daskalakis and Weinberg [22], give a PTAS, however instead of an LP-based approach they provide some structural probabilistic properties and run a modified, threshold-like VCG mechanism for most items and leave the remaining (constant number) to be handled by known approximate results like the ones of Daskalakis and Weinberg [22]. In earlier works, Chawla et al. [15] and Alaei [1] had provided (inferior) constant factor approximations for *unit-demand* bidders or general matroid constraints. The former paper deploys sequential posted pricing and uses a reduction to a single-dimensional optimal revenue problem, while the latter uses convex-

[10]The term has been used in [25] to state the difference between models where the values are given *explicitly* in the input of the algorithm, i.e. in the form of a list holding every point in the support and the corresponding probability, and models where this is given *implicitly* by just providing the marginal distribution for every item. The latter is the approach followed in the present thesis. As Daskalakis et al. [25] argue, the explicit model is clearly not the natural choice when we want to study continuous valuation priors and, settings in general that have some kind of structure that allows for more succinct representations than exhaustive listing all values. Notice also the exponential size requirement in the input of the mechanism under the explicit model.

programming relaxations. LP relaxations and rounding schemes have been also used by Bhattacharya et al. [7] to give constant approximations deploying sequential posted pricing, but without the unit-demand constraint. Correlation is allowed but MHR is required. If the MHR condition is relaxed, then they can guarantee logarithmic approximations. If the support of the priors of each bidder is discrete, they can show a 4-approximation by utilizing a sequential all-pay mechanism.

Unit-demand valuations Although the main topic of this thesis is on additive valuations, we briefly discuss some important results in the unit-demand front. In the economics community, Thanassoulis [75] was the first to clearly study the multidimensional revenue maximization problem specifically for unit-demand bidders, investigating whether a previous result of Riley and Zeckhauser [68] about deterministic optimality still holds for multiple goods. More recently, Pavlov [66] studied the unit-demand setting for two items with a uniform joint distribution over a triangular domain, giving exact descriptions of the optimal auction. In doing so, he showed something more general: the optimal auction must be piecewise linear, with 2 or 3 different regions (excluding the zero, non-allocation region). Notice that these results show that the optimal auction (for a uniform distribution) over the unit square I^2 is deterministic, while the one over the triangle with edge length 2 has a randomized component. Elements of this piece-wise linearity can be found also in the topological characterizations of Manelli and Vincent [54] (for additive valuations).

Computational complexity Daskalakis et al. [26] show that even for a single buyer and independent (but not identical) valuations with a finite support of size 2, it is #P-hard to compute exactly the allocation function of an optimal auction (see Corollary 1 in their text, which is in fact a special case of the more general Theorem 1). Therefore approximation is necessary in non-i.i.d. settings. However this does not exclude the possibility of a PTAS. Recall that Cai and Huang [10] and Daskalakis and Weinberg [22] have already given PTAS for the i.i.d. setting. The general idea of the techniques of Daskalakis et al. [26] is the relaxation of an LP describing the problem, the dual of which can then be interpreted as a flow problem. This hardness result strengthens even more the belief of the researchers working in the field that closed-form descriptions of optimal multidimensional mechanisms are, in general, not feasible.

Price Schedules We must, once more, make clear that the problem we are dealing with here involves optimization within the general space of truthful, IR auctions. There is a natural restriction of the problem when optimization is done only among the space of *deterministic* mechanism. Notable work in the economics community includes that of Armstrong [4]. Regarding computational complexity, for the unit-demand version of the problem, known as *item pricing*, Chen et al. [17] show that although it is polynomial time tractable for distributions of support of size 2, the decision version of the problem is NP-complete for larger supports, and remains NP-complete even for identical distributions.

Experimental results Chu et al. [19] perform numerical analysis over various, non-identical distributions for the single-buyer multiple-goods settings, wanting to see how well pricing according to the *cardinality* of the bundle performs when compared to the optimal price schedule (which in general can be exponential in size of its description) or to pricing the full bundle. Their experimental results are generally positive, and when the number of items increases or the marginal costs decrease, then the performance of their selling mechanism becomes even better.

Chapter 3

Duality

In this chapter we present our duality-theory framework for optimal mechanism design with additive valuations. It has two key characteristics. First, its generality: it can be formulated for any number of buyers and goods, and any kind of correlated joint probability prior over the bidders' type space, under only mild continuity assumptions. And secondly, its simplicity: the formulation resembles traditional linear programming (LP) duality, and one can readily plug-in the distributional priors and get closed form expressions for the objectives and the constraints. Furthermore, tools analogous to LP weak duality and complementarity can be shown to hold. Of course, here the optimization programs are no more discrete and finite, but instead involve continuous functions and their derivatives, and thus the results from classical LP duality don't come for free. As a matter of fact, the entire chapter can be seen as an effort to rigorously generalize the spirit and power of the theory of LP duality to such functional settings, while maintaining its main attractive features.

In Section 3.1 we present the formulation of the primal and dual programs, and discuss how these are linked to the revenue maximization problem. In Section 3.2 the tools of weak duality and (approximate) complementarity are stated and proven. Section 3.3 is dedicated to discussing some subtle points, like the importance and the effect of the initial relaxation of the convexity requirement when forming the primal program, as well as further relaxations and the application of the framework in unbounded domains.

3.1 The Formulation

In this chapter we will use our notation from Section 2.3: There are n bidders with additive valuations and m items, $x_{i,j} \in D_{i,j} = [L_{i,j}, H_{i,j}]$ denoting the value of player i for good j. The type profile $\mathbf{x}$ is drawn from a joint probability distribution F over $D = \prod_{i=1}^{n} \prod_{j=1}^{m} D_{i,j}$. The only assumption we will make for the distribution is that it has a density function f which is continuously differentiable[1].

[1] In fact, weaker assumptions like Lipschitz continuity or even just *absolute continuity* with respect to all its components would be enough for our purposes: we just need some minimal condition in order

3.1.1 Primal

Following the exposition in Section 2.3.1, finding the revenue maximizing mechanism is equivalent to maximizing the expectation (under distribution F) of the sum $\sum_{i=1}^{n} \nabla u_i(\mathbf{x}) \cdot \mathbf{x}_i - u_i(\mathbf{x})$ over all feasible utility functions $u_i : D \longrightarrow \mathbb{R}_+$. Feasibility here means that these functions must be *convex* and that their partial derivatives must be in $[0, 1]$, since they correspond to allocation probabilities. These requirements come from the fact that we restrict our search only within *truthful* mechanisms (see Theorem 2.2). Recall though that this is without loss to the revenue maximization objective, due to the Revelation Principle (Theorem 2.1)). Then, the optimal auction $\mathcal{M} = (\mathbf{a}, \mathbf{p})$ can be recovered easily by the utility functions: the allocation rule is simply given by $a_{i,j}(\mathbf{x}) = \frac{\partial u_i(\mathbf{x})}{\partial x_{i,j}}$ and the payment extracted from player i is $p_i(\mathbf{x}) = \nabla u_i(\mathbf{x}) \cdot \mathbf{x}_i - u_i(\mathbf{x})$.

We will relax this original problem by replacing the convexity constraint by the much milder constraint of *absolute continuity*—absolute continuity allows us to express functions as integrals of their derivatives[2]. We can restate this as follows: truthfulness in general imposes two conditions on the solution of allocating the items to bidders (see Theorem 2.2); the first condition is that the utility is convex; the second one is that the allocations must be gradients of the utility.

It seems that in many interesting cases, including the important Myersonian case of one item and regular distributions (Definition 2.6), when we optimize revenue the convexity constraint is essentially redundant: dropping it has no effect on the actual optimal value of the objective. However, there are cases in which this is not true and convexity is essential. We give an in-depth discussion of this topic, along with counterexamples, in Section 3.3.1. Notice though that in any case, the optimal value of the relaxed problem will always provide an upper bound on the optimal revenue, which is very important since the critical Weak Duality lemmas of Section 3.2 will be valid for both problems, regardless of whether the solution is convex or not.

So, we are going to refer to the following optimization problem as our *primal program*:

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to guarantee that basic integration operations can be performed. In particular, these are *integration by parts* and the *(second) fundamental theorem of calculus*. For more details on how all these conditions relate to each other and why they are enough, we refer to any serious introductory text in Measure Theory, e.g. these of Tao [74, Section 1.6] or Stein and Shakarchi [73, Section 3.2].

[2] Every convex function is Lipschitz continuous and thus also absolutely continuous with respect to all its variables (for a proof of this and a more thorough discussion see Rockafellar [72, Chapter 10]). See also Footnote 1 above.

$$\text{maximize} \qquad \sum_{i=1}^{n} \int_D \nabla u_i(\mathbf{x}) \cdot \mathbf{x}_i - u_i(\mathbf{x})\, dF(\mathbf{x}) \tag{3.1}$$

over the space of absolutely continuous[3] functions $u_i : D \longrightarrow \mathbb{R}_+$ having the properties

$$\sum_{i=1}^{n} \frac{\partial u_i(\mathbf{x})}{\partial x_{i,j}} \leq 1 \qquad (z_j(\mathbf{x}))$$

$$\frac{\partial u_i(\mathbf{x})}{\partial x_{i,j}} \geq 0 \qquad (s_{i,j}(\mathbf{x}))$$

for all $i \in [n]$, $j \in [m]$ and a.e. $\mathbf{x} \in D$.

We have appropriately labelled the constraints on the partial derivatives in order to use them in the duality formulation of the next section. Also, maintaining the spirit of traditional LP theory, we will feel free to refer to the u_i's of this program as the *primal variables*. A *feasible solution* is a set of variables u_i that satisfy all the necessary constraints of the program. We will refer to the value of the objective function (3.1) of the program evaluated on a specific feasible solution as the *value* of that solution. If a particular feasible solution is also a maximizer of the value of the program, it will be called *optimal*.

3.1.2 Dual

Motivated by traditional LP duality theory, we develop a duality framework that can be applied to the problem of designing auctions with optimal expected revenue. By interpreting the derivatives as differences, we can view the primal Program (3.1) as an (infinite) linear program and we can find its dual. The variables of the primal linear program are the values of the functions $u_i(\mathbf{x})$. The labels $(z_j(\mathbf{x}))$ and $(s_{i,j}(\mathbf{x}))$ on the constraints of Program (3.1) are the analogue of the dual variables of a linear program.

To find its dual program, we first rewrite the primal objective function in terms of the u_i's instead of their derivatives. In particular, by integration by parts we have

$$\int_D \frac{\partial u_i(\mathbf{x})}{\partial x_{i,j}} x_{i,j} f(\mathbf{x})\, d\mathbf{x} = \int_{D_{-i,j}} [u_i(\mathbf{x}) x_{i,j} f(\mathbf{x})]_{x_{i,j}=L_{i,j}}^{x_{i,j}=H_{i,j}}\, d\mathbf{x}_{-i,j} - \int_D u_i(\mathbf{x}) \frac{\partial (x_{i,j} f(\mathbf{x}))}{\partial x_{i,j}}\, d\mathbf{x}$$
$$= \int_{D_{-i,j}} [u_i(\mathbf{x}) x_{i,j} f(\mathbf{x})]_{x_{i,j}=L_{i,j}}^{x_{i,j}=H_{i,j}}\, d\mathbf{x}_{-i,j} - \int_D u_i(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} - \int_D u_i(\mathbf{x}) x_{i,j} \frac{\partial f(\mathbf{x})}{\partial x_{i,j}}\, d\mathbf{x}$$

[3] By this, we mean that every function u_i is absolutely continuous with respect to *all its components*, i.e. when viewed as a function $u_i(\;\cdot\;, \mathbf{x}_{-i,j}) : D_{i,j} \longrightarrow \mathbb{R}_+$, for all $i \in [n]$, $j \in [m]$ and fixed $\mathbf{x}_{-i,j} \in D_{-i,j}$. In fact, absolute continuity is needed only for the variables in $\mathbf{x}_i$, and simple continuity (or just integrability) for the remaining coordinates is enough. Finally, notice that given the constraints of the primal program on the partial derivatives, this means that u_i is not just absolutely continuous with respect to its i-th components, but also *m-Lipschitz continuous*. See also Footnote 1.

to rewrite the objective of the primal program as

$$\sum_{i=1}^{n}\int_{D}\left(\nabla u_i(\mathbf{x})\cdot\mathbf{x}-u_i(\mathbf{x})\right)\,dF(\mathbf{x}) =$$
$$\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{D_{-i,j}} H_{i,j}\,u_i(H_{i,j},\mathbf{x}_{-i,j})\,f(H_{i,j},\mathbf{x}_{-i,j})\,dx_{-i,j}$$
$$-\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{D_{-i,j}} L_{i,j}\,u_i(L_{i,j},\mathbf{x}_{-i,j})\,f(L_{i,j},\mathbf{x}_{-i,j})\,dx_{-i,j}$$
$$-\sum_{i=1}^{n}\int_{D} u_i(\mathbf{x})\left((m+1)f(\mathbf{x})+\mathbf{x}_i\cdot\nabla_i f(\mathbf{x})\right)\,d\mathbf{x}. \tag{3.2}$$

Notice that some of the above expressions make sense only for bounded domains (i.e., when $H_{i,j}$ is *not* infinite), but it is possible to extend the duality framework to unbounded domains, by carefully replacing these expressions with their limits when they exist or by appropriately truncating the probability distributions. For completeness and future reference we provide a treatment of the general case in Section 3.3.2.

To find the dual program, we have to take extra care on the boundaries of the domain, since the derivatives correspond to differences from which one term is missing (the one that corresponds to the variables outside the domain). Inside the domain, the dual constraint that corresponds to the primal variable $u_i(\mathbf{x})$ is

$$\sum_{j=1}^{m}\frac{\partial z_j(\mathbf{x})}{\partial x_{i,j}} \leq (m+1)f(\mathbf{x})+\mathbf{x}_i\cdot\nabla_i f(\mathbf{x}).$$

So, the *dual program* that we propose is

$$\text{minimize} \qquad \sum_{j=1}^{m}\int_{D} z_j(\mathbf{x})\,d\mathbf{x} \tag{3.3}$$

over the space of absolutely continuous functions $z_j, s_{i,j} : D \longrightarrow \mathbb{R}_+$ having the properties

$$\sum_{j=1}^{m}\left(\frac{\partial z_j(\mathbf{x})}{\partial x_{i,j}}-\frac{\partial s_{i,j}(\mathbf{x})}{\partial x_{i,j}}\right) \leq (m+1)f(\mathbf{x})+\mathbf{x}_i\cdot\nabla_i f(\mathbf{x}) \qquad (u_i(\mathbf{x}))$$
$$z_j(L_{i,j},\mathbf{x}_{-i,j})-s_{i,j}(L_{i,j},\mathbf{x}_{-i,j}) \leq L_{i,j}f(L_{i,j},\mathbf{x}_{-i,j}) \qquad (u_i(L_{i,j},\mathbf{x}_{-i,j}))$$
$$z_j(H_{i,j},\mathbf{x}_{-i,j})-s_{i,j}(H_{i,j},\mathbf{x}_{-i,j}) \geq H_{i,j}f(H_{i,j},\mathbf{x}_{-i,j}) \qquad (u_i(H_{i,j},\mathbf{x}_{-i,j}))$$

for all $i \in [n]$, $j \in [m]$ and a.e. $\mathbf{x} \in D$.

Again, we have labelled the constraints of this dual program so that they match appropriately to the corresponding primal variables of (3.1).

The above intuitive derivation of this dual is used only for illustration and for explaining how we came up with it. None of the results relies on the actual way of coming up with the dual program. However, the derivation is useful for intuition and for suggesting traditional linear programming machinery for these infinite systems; for example, although we don't directly use any results from the theory of linear programming duality, we are motivated by it to prove similar connections between our primal and dual programs.

One can interpret this dual as follows: For the sake of clarity, assume a single-bidder and drop the $s_{i,j}$ constraints; we seek m functions z_j defined inside the hyperrectangle $[L_1, H_1] \times \cdots \times [L_m, H_m]$ such that

- in the j-th direction, function z_j starts at value (at most) $L_j f(L_j, \mathbf{x}_{-j})$ and ends at value (at least) $H_j f(H_j, \mathbf{x}_{-j})$; this must hold for all $\mathbf{x}_{-j}$.
- at every point of the domain, the sum of the derivatives of functions z_j cannot exceed $(m+1)f(\mathbf{x}) + \mathbf{x} \cdot \nabla f(\mathbf{x})$.
- the sum of the integrals of these functions is minimized.

A visual interpretation of this for two and three uniformly distributed goods can be seen in Figure 4.1.

Let us also mention parenthetically that one can derive Myerson's results by selecting as variables not the utilities $u_i(\mathbf{x})$, but their *derivatives* (see [58]). In fact, since the allocation constraints involve exactly the derivatives, this is the natural choice of primal variables. Unfortunately though, such an approach does not seem to work for more than one item because the derivatives are not independent functions. If we treat them as independent, we lose the power of the gradient constraint.

3.2 Weak Duality and Complementarity

The way that we derived the dual system does not yet provide any rigorous connection with the original primal system. We now prove that this is indeed a weak dual, in the sense that the value of the dual minimization program cannot be less than the value of the primal program:

Lemma 3.1 (Weak Duality). *The value of every feasible solution of the primal Program* (3.1) *does not exceed the value of any feasible solution of the dual Program* (3.3).

Proof. The proof is essentially a straightforward adaptation of the proof of traditional weak duality for finite linear programs. Take a pair of feasible solutions for the primal and the dual programs and consider the difference between the dual objective (3.3) and the primal objective (3.2):

$$\sum_{j=1}^{m}\int_D z_j(\mathbf{x})\,d\mathbf{x} + \sum_{i\in[n]}\int_D u_i(\mathbf{x})\,\left((m+1)f(\mathbf{x}) + \mathbf{x}_i\cdot\nabla_i f(\mathbf{x})\right)\,d\mathbf{x}$$
$$-\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{D_{-i,j}} H_{i,j}\,u_i(H_{i,j},\mathbf{x}_{-i,j})\,f(H_{i,j},\mathbf{x}_{-i,j})\,d\mathbf{x}_{-i,j}$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{D_{-i,j}} L_{i,j}\,u_i(L_{i,j},\mathbf{x}_{-i,j})\,f(L_{i,j},\mathbf{x}_{-i,j})\,d\mathbf{x}_{-i,j}$$

Using the constraints of the programs, the first two terms of this expression can be bounded from below by

$$\sum_{j=1}^{m}\int_D z_j(\mathbf{x})\sum_{i=1}^{n}\frac{\partial u_i(\mathbf{x})}{\partial x_{i,j}}\,d\mathbf{x} - \sum_{i=1}^{n}\sum_{j=1}^{m}\int_D s_{i,j}(\mathbf{x})\sum_{i=1}^{n}\frac{\partial u_i(\mathbf{x})}{\partial x_{i,j}}\,d\mathbf{x}$$
$$+\sum_{i=1}^{n}\int_D u_i(\mathbf{x})\left(\sum_{j=1}^{m}\frac{\partial z_j(\mathbf{x})}{\partial x_{i,j}} - \sum_{j=1}^{m}\frac{\partial s_{i,j}(\mathbf{x})}{\partial x_{i,j}}\right)\,d\mathbf{x}$$

which equals

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\int_D \frac{\partial\left[(z_j(\mathbf{x})-s_{i,j}(\mathbf{x}))u_i(\mathbf{x})\right]}{\partial x_{i,j}}\,d\mathbf{x}.$$

Similarly, the other two terms of the expression can be bounded from below by

$$-\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{D_{-i,j}} u_i(H_{i,j},\mathbf{x}_{-i,j})\,\left(z_j(H_{i,j},\mathbf{x}_{-i,j}) - s_{i,j}(H_{i,j},\mathbf{x}_{-i,j})\right)\,d\mathbf{x}_{-i,j}$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{D_{-i,j}} u_i(L_{i,j},\mathbf{x}_{-i,j})\,\left(z_j(L_{i,j},\mathbf{x}_{-i,j}) - s_{i,j}(L_{i,j},\mathbf{x}_{-i,j})\right)\,d\mathbf{x}_{-i,j}$$

and they cancel out the first two terms. Bringing everything together, the difference of the dual and primal objectives are bounded from below by zero. □

One can use weak duality to show optimality, in the same way that we do that in traditional LP settings: it suffices to have a pair of feasible primal-dual solutions that have the same value. However powerful and straightforward this seems, sometimes computing the value of a primal or dual solution might not be easy at all or the optimal solutions may not even be expressible in a closed form. For example, that is the situation for the case of uniform distributions as we'll see in Chapter 3. Thus, it is essential to develop additional techniques that can be utilized in a more indirect and less explicitly constructive way to prove optimality: a useful tool towards that end is through *complementarity* (see e.g. [76]). In fact, we will prove a more powerful version than a simple generalization of traditional LP complementarity to continuous functions, which will allow us later to discretize the domain and consider *approximate solutions* as well. Specifically, instead of requiring the product of primal and corresponding dual constraints to be *exactly zero*, we generalize it to be bounded above by a constant:

Lemma 3.2 (Complementarity)**.** *Suppose that* $u_i(\mathbf{x})$ *is a primal feasible solution and* $z_j(\mathbf{x}), s_{i,j}(\mathbf{x})$ *is a dual feasible solution. Fix some parameter* $\varepsilon \geq 0$*. If the following complementarity constraints hold for all* $i \in [n]$*,* $j \in [m]$ *and a.e.* $\mathbf{x} \in D$*,*

$$u_i(\mathbf{x}) \cdot \left((m+1)f(\mathbf{x}) + \mathbf{x}_i \cdot \nabla_i f(\mathbf{x}) - \sum_{j=1}^{m} \frac{\partial z_j(\mathbf{x})}{\partial x_{i,j}} + \sum_{j=1}^{m} \frac{\partial s_{i,j}(\mathbf{x})}{\partial x_{i,j}}\right) \leq \varepsilon f(\mathbf{x})$$

$$u_i(L_{i,j}, \mathbf{x}_{-i,j}) \cdot (L_{i,j} f(L_{i,j}, \mathbf{x}_{-i,j}) - z_j(L_{i,j}, \mathbf{x}_{-i,j}) + s_{i,j}(L_{i,j}, \mathbf{x}_{-i,j})) \leq \varepsilon f(L_{i,j}, \mathbf{x}_{-i,j})$$

$$u_i(H_{i,j}, \mathbf{x}_{-i,j}) \cdot (z_j(H_{i,j}, \mathbf{x}_{-i,j}) - s_{i,j}(H_{i,j}, \mathbf{x}_{-i,j}) - H_{i,j} f(H_{i,j}, \mathbf{x}_{-i,j})) \leq \varepsilon f(H_{i,j}, \mathbf{x}_{-i,j})$$

$$z_j(\mathbf{x}) \cdot \left(1 - \sum_{i=1}^{n} \frac{\partial u_i(\mathbf{x})}{\partial x_{i,j}}\right) \leq \varepsilon f(\mathbf{x}),$$

then the primal and dual objective values differ by at most $(n + m + 2nm)\varepsilon$*. In particular, if the conditions are satisfied with* $\varepsilon = 0$*, both solutions are optimal.*

A pair of primal-dual solutions that satisfies the conditions of the lemma for some $\varepsilon > 0$ will be called ε-*complementary*. If this happens with $\varepsilon = 0$, then they will be simply called *complementary.*

Proof. We take the sum of all complementarity constraints and integrate in the domain:

$$\sum_{i=1}^{n} \int_D u_i(\mathbf{x}) \cdot \left((m+1)f(\mathbf{x}) + \mathbf{x}_i \cdot \nabla_i f(\mathbf{x}) - \sum_{j=1}^{m} \frac{\partial z_j(\mathbf{x})}{\partial x_{i,j}} + \sum_{j=1}^{m} \frac{\partial s_{i,j}(\mathbf{x})}{\partial x_{i,j}}\right) d\mathbf{x}$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{D_{-i,j}} u_i(L_{i,j}, \mathbf{x}_{-i,j}) \cdot (L_{i,j} f(L_{i,j}, \mathbf{x}_{-i,j}) - z_j(L_{i,j}, \mathbf{x}_{-i,j}) + s_{i,j}(L_{i,j}, \mathbf{x}_{-i,j}))\, d\mathbf{x}_{-i,j}$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{D_{-i,j}} u_i(H_{i,j}, \mathbf{x}_{-i,j}) \cdot (z_j(H_{i,j}, \mathbf{x}_{-i,j}) - s_{i,j}(H_{i,j}, \mathbf{x}_{-i,j}) - H_{i,j} f(H_{i,j}, \mathbf{x}_{-i,j}))\, d\mathbf{x}_{-i,j}$$

$$+ \sum_{j=1}^{m} \int_D z_j(\mathbf{x}) \cdot \left(1 - \sum_{i=1}^{n} \frac{\partial u_i(\mathbf{x})}{\partial x_{i,j}}\right) dx \leq (n + m + 2nm)\varepsilon$$

It suffices to notice that, by using the same transformations that we used to prove the Weak Duality Lemma 3.1, the left hand side is equal to the dual objective minus the primal objective. □

3.2.1 Discussion

The above tools now allow us to approach the problem of maximizing revenue in many different ways:

1. Starting from a specific truthful auction, that is with a *convex* feasible solution to the primal program, and then coming up with an explicit feasible dual solution
   (a) whose value exactly matches the value of the primal solution; then, by weak duality, this auction is optimal.

(b) whose value is an α-approximation of the value of the primal solution; then, by weak duality again, this auction is α-approximate. In particular, the value of the dual solution is an upper bound to the optimal revenue achieved by any auction, since the primal program is a relaxation to the original revenue maximization problem (by having dropped convexity).

2. Starting from a specific truthful auction and then showing, possibly in a non-constructive way, the existence of

   (a) a feasible dual solution which is complementary to the primal; then this auction is optimal.

   (b) a *family* of feasible dual solutions, indexed by a real parameter $\varepsilon > 0$, which is ε-complementary to the primal; then, taking the limit as $\varepsilon \to 0$, proves that again this auction is optimal. Notice that this approach would also work for point (1b) above, by taking $\alpha \to 1$.

3. Doing all the above for a feasible primal solution $\mathbf{u}$ which is however *not* convex. This means, that we may have managed to solve, or approximate, the primal-dual system, but this didn't give rise to a *feasible* auction, due to our initial relaxation of the primal program by dropping the convexity constraint. Then, we may be able to get an actual feasible auction with good performance, by transforming the primal solution $\mathbf{u}$ to a convex one $\mathbf{u}'$ which is "close" to $\mathbf{u}$. This *"convexification"* process can be viewed as the analogue of *rounding* a relaxed solution in traditional LP (see e.g. [76]).

All these techniques will be further demonstrated and used in the following chapters of this thesis.

3.2.2 Further Relaxations

The dual Program (3.3) is indeed powerful and straightforward enough to readily give us closed-form and manageable expressions for both the objective function and the constraints, for a wide range of distributional priors. However, further simplification would admittedly be very useful, given especially the notorious difficulty of multidimensional revenue optimization. Towards this direction, for all applications in the following chapters of this thesis, we will further relax the primal Program (3.1) by dropping the lower bound constraint on the derivatives, that is $\nabla u_i(\mathbf{x}) \geq \mathbf{0}$: effectively this means that the $s_{i,j}$ dual variables will not appear any more in the dual Program (3.3). Furthermore, these applications will involve a single buyer, so the tools of our duality framework can be simplified: the primal program becomes

$$\text{maximize} \qquad \int_D \nabla u(\mathbf{x}) \cdot \mathbf{x} - u(\mathbf{x})\, dF(\mathbf{x}) \tag{3.4}$$

over the space of absolutely continuous functions $u : D \longrightarrow \mathbb{R}_+$ with

$$\frac{\partial u(\mathbf{x})}{\partial x_j} \leq 1, \qquad (z_j(\mathbf{x}))$$

the dual

$$\text{minimize} \qquad \sum_{j=1}^{m} \int_D z_j(\mathbf{x})\, d\mathbf{x} \qquad (3.5)$$

over the space of absolutely continuous functions $z_j : D \longrightarrow \mathbb{R}_+$ having the properties

$$\sum_{j=1}^{m} \frac{\partial z_j(\mathbf{x})}{\partial x_{i,j}} \leq (m+1)f(\mathbf{x}) + \mathbf{x} \cdot \nabla f(\mathbf{x}) \qquad (u(\mathbf{x}))$$

$$z_j(L_j, \mathbf{x}_{-j}) \leq L_{i,j} f(L_j, \mathbf{x}_{-j}) \qquad (u(L_j, \mathbf{x}_{-j}))$$

$$z_j(H_j, \mathbf{x}_{-j}) \geq H_j f(H_j, \mathbf{x}_{-j}), \qquad (u(H_j, \mathbf{x}_{-j}))$$

and finally the complementarity constraints are simplified to

Lemma 3.3 (Complementarity for a single bidder)**.** *Suppose that $u(\mathbf{x})$, $z_j(\mathbf{x})$ is a pair of feasible primal-dual solutions. Fix some parameter $\varepsilon \geq 0$. If the following complementarity constraints hold for all $j \in [m]$ and a.e. $\mathbf{x} \in D$,*

$$u(\mathbf{x}) \cdot \left((m+1)f(\mathbf{x}) + \mathbf{x} \cdot \nabla f(\mathbf{x}) - \sum_{j=1}^{m} \frac{\partial z_j(\mathbf{x})}{\partial x_{i,j}} \right) \leq \varepsilon f(\mathbf{x})$$

$$u(L_j, \mathbf{x}_{-j}) \cdot (L_j f(L_j, \mathbf{x}_{-j}) - z_j(L_j, \mathbf{x}_{-j})) \leq \varepsilon f(L_{i,j}, \mathbf{x}_{-j})$$

$$u(H_j, \mathbf{x}_{-j}) \cdot (z_j(H_j, \mathbf{x}_{-j}) - H_j f(H_j, \mathbf{x}_{-j})) \leq \varepsilon f(H_j, \mathbf{x}_{-j})$$

$$z_j(\mathbf{x}) \cdot \left(1 - \frac{\partial u(\mathbf{x})}{\partial x_j} \right) \leq \varepsilon f(\mathbf{x}),$$

then the primal and dual objective values differ by at most $(3m+1)\varepsilon$. In particular, if the conditions are satisfied with $\varepsilon = 0$, both solutions are optimal.

There provably exist cases where this relaxation comes with a loss to the optimization objective, even for the simplest case of a single buyer and only one good for sale: in Section 3.3.1 we explicitly give an example of a probability distribution over $[0, 1]$ for which the dual variable $s_{1,1}$ is needed for strong duality. However, this distribution is *irregular* (Definition 2.6)). As a matter of fact, all the specific examples we will deal with from now on in this thesis will involve a single buyer (but still many items) and independent distributions that demonstrate some forms of regularity, and it seems that this is enough to cause no loss under the relaxation of the positive derivatives constraint. We must though point out here, that we believe this *not* to be the case in general: we conjecture that the $s_{i,j}$'s are necessary, even for regular independent distributions, if *multiple-bidder* settings are considered.

In any case, we chose in this chapter to state our primal-dual framework in its most general form and full power, involving many players, multiple items and arbitrarily correlated, possibly irregular, distributional priors, since we are confident that this might prove useful in the future for attacking particular auction problems in such general settings. Our priority though in this thesis, after the formulation of the framework, would be to tackle standard, long standing open problems in the area of multidimensional auctions, and this firstly requires dealing with the single-buyer case for which the results and structural understanding have been very limited so far.

3.3 Fine Points

3.3.1 Convexity

In this section we discuss in greater depth the convexity constraint of the utility functions. For clarity, we focus on the simple case of one bidder and a single item. For that case we show that the convexity constraint is not necessary when item value is drawn from a distribution that satisfies a *regularity* condition (see (3.6)). And in the opposite direction, we exhibit an example of a distribution which does not satisfy the regularity condition and for which the convexity constraint cannot be dropped without affecting optimality. Later on, in Section 5.6.3 we will see another case where convexity is necessary, this time for regular distributions but for *two* goods.

The primal program (3.1) (taking into consideration (3.2)) for this case is

$$\max_u \quad u(H)Hf(H) - u(L)Lf(L) - \int_L^H u(x)(f(x) + (xf(x))')\,dx$$

subject to

$$\begin{aligned} u'(x) &\leq 1 && (z(x)) \\ u'(x) &\geq 0 && (s(x)) \\ u''(x) &\geq 0 && (w(x)) \\ u(x) &\geq 0. \end{aligned}$$

Notice that there is no reason to include $u(L) = 0$ since this holds for the optimal solution anyway; that is because if $u(x)$ and $u(x) - c$ are both feasible solutions and c is a positive constant, then the corresponding objectives differ by $cHf(H) - cLf(L) - \int_L^H c(f(x) + (xf(x))')\,dx = -c(H \cdot F(H) - L \cdot F(L)) < 0$; this shows that the optimal solution has $u(x) = 0$ for some x. See also the discussion about this in Section 3.3.4.

In our treatment of the subject in this thesis, we are dropping the constraints labelled by the dual variables $w(x)$ (convexity) and for the most part we will also drop the ones corresponding to $s(x)$ (nonnegative derivatives). We do that to keep the primal and dual systems simple. More importantly, there is a strong reason for ignoring the constraints corresponding to $w(x)$ for multi-parameter domains: the convexity constraints $\nabla^2 u(x) \succeq 0$ (that is, the Hessian of u being *positive semidefinite*, see [72]) are not linear in u (unlike the one-dimensional case, in which the constraint $u''(x) \geq 0$ is linear in u).

In the rest of this subsection, we investigate when the simplified systems are optimal. We first give the dual of the above complete, non-relaxed primal[4]:

$$\max z, s, w \int_L^H z(x)\, dx$$

subject to

$$\begin{aligned} z'(x) - s'(x) + w''(x) &\geq f(x) + (xf(x))' && (u(x)) \\ z(H) - s(H) - w(H) &\geq Hf(H) && (u(H)) \\ z(L) - s(L) - w(L) &\leq Lf(L) && (u(L)) \end{aligned}$$

For a large class of distributions, the primal constraints with labels $s(x)$ and $w(x)$ are non-essential; that is, if we set the dual variables $s(x)$ and $w(x)$ to zero, the optimal solution is not affected. In particular, let us drop the $s(x)$ and $w(x)$ constraints, and consider the distributions which satisfy

$$f(x) + (xf(x))' \geq 0. \tag{3.6}$$

This condition is equivalent to $F(x) + xf(x) - 1 = -(x(1 - F(x)))'$ being (weakly) increasing. Equivalently, the *revenue curve* $R(x) = x(1 - F(x))$, which gives the

[4]The dual of a linear program with derivatives can be computed in a straightforward way, in the spirit of the derivation in Lemma 3.1. Instead of giving the rules, we simply give an illustrating example. If the primal program with variable $h(x)$ is

$$\max_h \int_D h(x)\, \gamma(x)\, dx$$

subject to

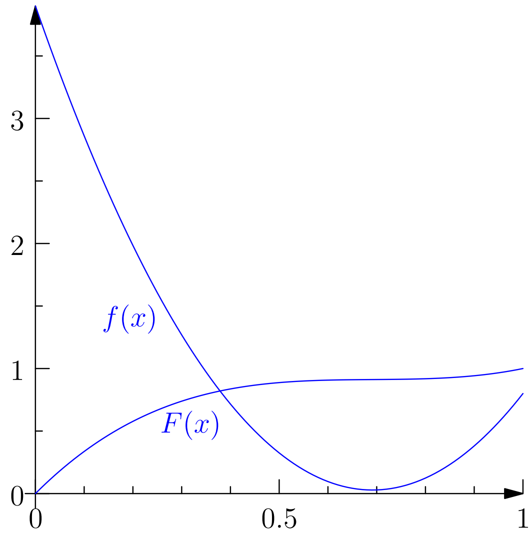
$$\begin{aligned} \alpha_0(x)h(x) + \alpha_1(x)h'(x) + \alpha_2(x)h''(x) &\leq \beta(x) && (g(x)) \\ h(x) &\geq 0, \end{aligned}$$

its dual program with variable $g(x)$ is

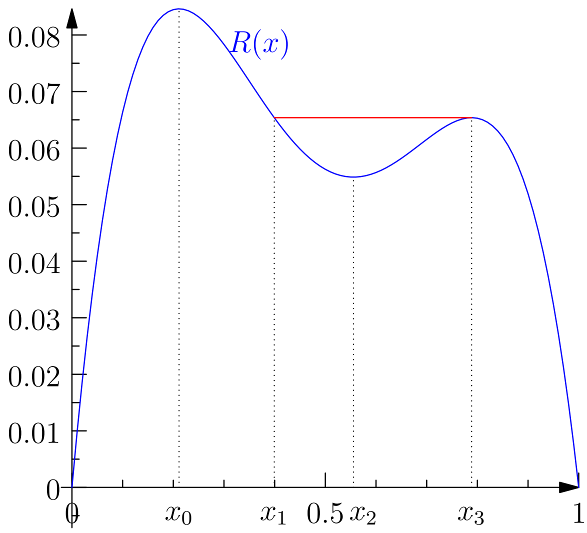
$$\min_g \int_D g(x)\, \beta(x)\, dx$$

subject to

$$\begin{aligned} \alpha_0(x)g(x) - (\alpha_1(x)g(x))' + (\alpha_2(x)g(x))'' &\geq \gamma(x) && (h(x)) \\ g(x) &\geq 0. \end{aligned}$$



(a) The distribution



(b) The revenue curve $R(x) = x(1 - F(x))$

Figure 3.1: The probability distribution in Figure 3.1a is not regular and does not have concave revenue curve, i.e. $F(x) + xf(x) - 1$ is not monotone. Its revenue curve is shown in Figure 3.1b. The points x_0, x_2, and x_3 are extrema; the point x_1 has the same revenue with x_3.

revenue for the deterministic mechanism that sells the item at (reserve) price x, is *not* concave. This condition is very similar to the regularity condition of Myerson [58] (Definition 2.6). When (3.6) holds, let x_0 be such that $F(x_0) + x_0 f(x_0) - 1 = 0$ (x_0 is effectively Myerson's *virtual* price); the optimal primal and dual solutions are $u(x) = \max\{0, x - x_0\}$ and $z(x) = \max\{0, F(x) + xf(x) - 1\}$, or equivalently

$$z(x) = \begin{cases} 0 & x \leq x_0 \\ F(x) + xf(x) - 1 & x \geq x_0. \end{cases}$$

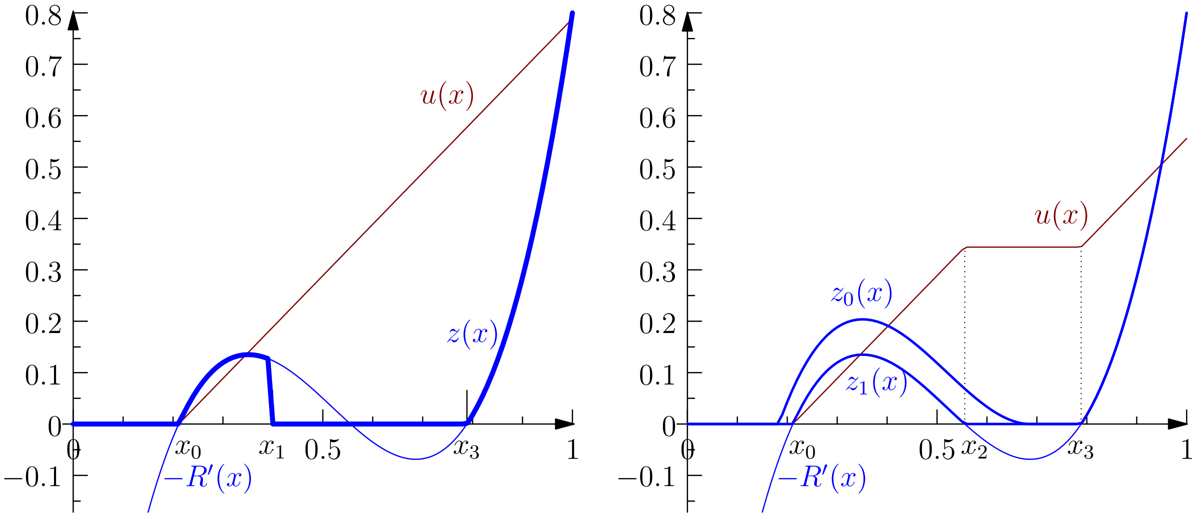
It is straightforward to check that the above conditions satisfy the primal and dual constraints and that they are complementary (that is, they satisfy Lemma 3.2 with $\varepsilon = 0$); therefore they are optimal.

Let us now consider the case when condition (3.6) does *not* hold. Figures 3.1 and 3.2 show an example, for the probability distribution with cumulative function

$$F(x) = 1 - (1 - x)(1 + x(2.7x - 2.9))$$

over the unit interval I, for which the convexity constraint is necessary to get optimal solutions. The probability distribution on the left of Figure 3.1 is *not* regular and its revenue function $R(x) = x(1 - F(x))$ is not concave; equivalently, $-R'(x) = F(x) + xf(x) - 1$ is not increasing. The revenue curve is shown on the right of the figure. The points x_0, x_2, and x_3 are extrema; the point x_1 induces the same revenue with x_3.

Figure 3.2 shows the optimal solutions. The optimal primal solution is $u(x) = \max(0, x - x_0)$ and corresponds to the deterministic selling mechanism with reserve price x_0. The optimal dual solution is $z(x)$ which is equal to $F(x) + xf(x) - 1$ in the



(a) Optimal solutions (b) Relaxed optimal solutions

Figure 3.2: Figure 3.2a shows the optimal solutions for the distribution of Figure 3.1. Figure 3.2b shows the optimal dual solution $z_1(x)$ and the corresponding optimal solution $u(x)$, when we drop the convexity constraint. Function $z_0(x)$ is the optimal dual solution when we drop both the convexity constraint and the constraint $u'(x) \geq 0$.

intervals $[x_0, x_1]$ and $[x_3, 1]$. The integral of $-R'(x)$ in $[x_1, x_3]$ is 0 (by the definition of x_1). The flattening of the $-R'(x)$ curve in this interval is similar to the *ironing process* of Myerson [58]. It follows that the integral of $z(x)$ in the interval $[x_0, 1]$ is equal to the integral of $-R'(x)$, and therefore equal to $x_0(1 - F(x_0))$, which is the optimal value.

This example illustrates that convexity is in some cases necessary to obtain optimal primal and dual solutions. The right part of Figure 3.2 shows the optimal dual solution $z_1(x) = \max\{0, -R'(x)\}$ and the corresponding primal solution when we drop the convexity constraint. To see that this is optimal, observe first that the dual solution with $z(x) = z_1(x)$ and $s(x) = \max\{0, R'(x)\}$ is feasible (in fact, it satisfies the $u(x)$ and $u(H)$ constraints *tightly*) and has optimal value

$$\begin{aligned}\int_0^1 z(x)\,dx &= \int_{x_0}^{x_2} -R'(x)\,dx + \int_{x_3}^{1} -R'(x)\,dx \\ &= R(x_0) - R(x_2) + R(x_3) - R(1) \\ &= R(x_0) - R(x_2) + R(x_3).\end{aligned}$$

It is straightforward to verify that the primal solution $u(x)$ has the same value, which shows that it is optimal. However, the primal solution is not convex and, furthermore, its value is strictly higher than the optimal one (because $R(x_3) > R(x_2)$). Therefore *the convexity constraint is in general essential to obtain the optimal solution.* Recall though that this is not the case for distributions that satisfy the regularity condition (3.6).

It is worth mentioning also that, if we drop *both* the convexity constraint as well as condition $u'(x) \geq 0$, we obtain an even worse (higher) dual solution. This is $z_0(x)$ depicted in the right part of Figure 3.2. Observe that in this case, the value of $z_0(x)$

must be positive when $-R'(x) = F(x) + xf(x) - 1$ is strictly decreasing (since its derivative must be negative and its value cannot become negative; there is no dual variable $s(x) = s_{1,1}(x)$ to absorb this effect).

Finally, we believe it is very interesting to notice the interpretation of the dual variable: $z(x)$ *is the negative derivative of the revenue curve.*

3.3.2 Unbounded Domains

As we mentioned in the presentation of the duality framework in Section 3.1.2, for it to make sense as-it-is we need the integrals in the basic transformation of the primal revenue-maximization objective in expression (3.2) to be well-defined. This is definitely the case when we have bounded domains, that is when the upper-boundary $H_{i,j}$ of each interval $D_{i,j} = [L_{i,j}, H_{i,j}]$ is finite: all integrals in (3.2) are finite and the integration-by-parts is valid. This of course includes the special case of uniform distributions which is one of the main topics of this thesis (Chapter 4). We will now discuss how one can still use this duality framework in cases where the domain D is *not* bounded.

For the sake of clarity, let us assume for the remaining of this section that we have a single bidder and that item values are i.i.d. from some distribution F with density f over an interval $[L, H]$, $L \geq 0$. First notice that, even for unbounded domains where $H = \infty$, the critical integral

$$\int_{D_{-j}} H_j\, u(H_j, \mathbf{x}_{-j})\, f(H_j, \mathbf{x}_{-j})\, d\mathbf{x}_{-j} = Hf(H) \int_{[L,H]^{m-1}} u(H, \mathbf{x}_{-j}) \prod_{l \neq j} f(x_l)\, d\mathbf{x}_{-j} \quad (3.7)$$

in (3.2) may still converge as $H \to \infty$. In such a case, the duality framework from Section 3.1.2 can be applied as it is: one just has to take the limit of $H \to \infty$ wherever H appears, and in particular the Weak Duality Lemma 3.1 is still valid if one replaces condition $z_j(H, \mathbf{x}_{-j}) \geq Hf(H, \mathbf{x}_{-j})$ by its natural limiting version of $\lim_{H\to\infty} \left(z_j(H, \mathbf{x}_{-j}) - Hf(H) \prod_{l\neq j} f(x_l) \right) \geq 0$. For example, a sufficient condition for distributions with unbounded support to still induce bounded values in (3.7) is to have *finite expectation*. This is a rather natural assumption to make and is standard for example in the works of Myerson [58] and Krishna [47]. To see why (3.7) is finite, it can be rewritten as $Hf(H) \cdot \mathbb{E}_{\mathbf{x}_{-j} \sim F^{m-1}} [u(H, \mathbf{x}_{-j})]$ and so, due to the derivatives constraint $\nabla u(\mathbf{x}) \leq \mathbf{1}_m$, it is upper-bounded by

$$Hf(H)\, \mathbb{E}_{\mathbf{x}_{-j} \sim F^{m-1}} \left[H + \sum_{l \neq j} x_l \right] = H^2 f(H) + (m-1) Hf(H)\, \mathbb{E}[X].$$

Now, if we take into consideration that any bounded-expectation distribution must have $f(x) = o(1/x^2)$ since $\mathbb{E}[X] = \int x f(x)\, dx$ must converge, then it is easy to see that this expression converges as $H \to \infty$, and in fact vanishes to zero.

However, this might not be true for distributions with *infinite* expectation, for example the equal revenue distribution (see Definition 2.8). In such a case, we can follow a different path in order to use our duality framework. One can take the *truncated* version of the distribution within a finite interval, that is consider the distribution $F_b(x) \equiv \frac{1}{F(b)}F(x)$ over the interval $[L, b]$ for any $b \geq L$, apply the duality theory framework in this finite case, and then study the behaviour as $b \to \infty$. As the following Theorem 3.1 shows, this process will be without loss.

One last remark before stating the theorem is that, whenever one deals with a specific case of the optimal revenue problem, he has to make sure that it is *well defined*, i.e. that $\textsc{Rev}(F) < \infty$ for the particular distributional priors F. This might seem obvious at first, and is indeed a very subtle point to consider, but let us note here that it is not the case for *any* probability distribution. For example, if we consider i.i.d. valuations from the Pareto distribution $f(x) = \frac{1}{2}x^{-3/2}$, $x \in [1, \infty)$, the expected (Myersonian) revenue by selling a single item at a price of t is $t(1 - F(t)) = t(1 - 1 + t^{-1/2}) = t^{1/2}$ which tends to infinity. Some simple sufficient conditions for bounded optimal revenue in the i.i.d. case where the valuations come from a product distribution F^m are the bounded expectation of the distribution F, since by IR one trivially gets the bound $\textsc{Rev}(F^m) \leq m\,\mathbb{E}[X]$, as well as the bounded revenue $\textsc{MRev}(F)$ for the single-item case, since from the work of Hart and Nisan [38] we know that there exists a constant $c > 0$ such that $\frac{c}{\log^2 m}\textsc{Rev}(F^m) \leq \textsc{SRev}(F^m)$, so we can get $\textsc{Rev}(F^m) \leq \frac{m \log^2 m}{c}\textsc{MRev}(F)$. The former condition is stronger. For example, the ER does not have finite expectation but it does induce a finite Myersonian revenue of 1.

Theorem 3.1. *Let F be a probability distribution over $[a, \infty)$, $a \geq 0$, such that $\textsc{Rev}(F^m) < \infty$. Then, if F_b denotes the truncation of F in $[a, b]$, $b \geq a$, and $\lim_{b\to\infty} \textsc{Rev}(F_b^m)$ converges, it must be that*

$$\lim_{b\to\infty} \textsc{Rev}(F_b^m) = \textsc{Rev}(F^m).$$

Proof. Let u be the utility function of an optimal selling mechanism when valuations are drawn i.i.d. from F. The restriction of u in $[a, b]$ is a valid utility function for the setting where valuations are drawn i.i.d. from F_b and also we know that $F(x) = F(b)F_b(x)$ for all $x \in [a, b]$. Combining these we get:

$$\begin{aligned}
\textsc{Rev}(F^m) &= \int_{[a,\infty)^m} \mathbf{x} \cdot \nabla u(\mathbf{x}) - u(\mathbf{x})\, dF^m(\mathbf{x}) \\
&= \int_{[a,b]^m} \mathbf{x} \cdot \nabla u(\mathbf{x}) - u(\mathbf{x})\, dF^m(\mathbf{x}) + \int_{[a,\infty)^m \setminus [a,b]^m} \mathbf{x} \cdot \nabla u(\mathbf{x}) - u(\mathbf{x})\, dF^m(\mathbf{x}) \\
&= F^m(b) \int_{[a,b]^m} \mathbf{x} \cdot \nabla u(\mathbf{x}) - u(\mathbf{x})\, dF_b^m(\mathbf{x}) + \int_{[a,\infty)^m \setminus [a,b]^m} \mathbf{x} \cdot \nabla u(\mathbf{x}) - u(\mathbf{x})\, dF^m(\mathbf{x}) \\
&\leq F^m(b)\textsc{Rev}(F_b^m) + \int_{[a,\infty)^m \setminus [a,b]^m} \mathbf{x} \cdot \nabla u(\mathbf{x}) - u(\mathbf{x})\, dF^m(\mathbf{x})
\end{aligned}$$

Next, for any $b \geq a$, let u_b be the utility function of an optimal selling mechanism when valuations are drawn i.i.d. from F_b. This utility function can be extended to a valid utility function $\overline{u}_b$ over the entire interval $[a, \infty)$ in the following way

$$\overline{u}_b(\mathbf{x}) = u_b(\gamma_b(\mathbf{x})) + (\mathbf{x} - \gamma_b(\mathbf{x})) \cdot \nabla u_b(\gamma_b(\mathbf{x})), \quad \mathbf{x} \in [a, \infty)^m,$$

where $\gamma_b(\mathbf{x})$ is the pointwise minimum of $\mathbf{x}$ and $(b)^m$, that is the m-dimensional vector whose j-th coordinate is $\min\{x_j, b\}$.

Since u_b is a convex function with partial derivatives in $[0, 1]$, so is the extended $\overline{u}_b$. This means that we immediately get

$$F^m(b)\textsc{Rev}(F_b^m) \leq \textsc{Rev}(F^m).$$

Now the theorem follows from the facts that $\lim_{b\to\infty} F^m(b) = 1$ and

$$\lim_{b\to\infty} \int_{[a,\infty)^m \setminus [a,b]^m} \mathbf{x} \cdot \nabla u(\mathbf{x}) - u(\mathbf{x}) \, dF^m(\mathbf{x}) = 0.$$

The last equality is due to the fact that $\int_{[a,\infty)^m} \mathbf{x} \cdot \nabla u(\mathbf{x}) - u(\mathbf{x}) \, dF^m(\mathbf{x})$ is bounded by assumption. □

3.3.3 Bayes-Nash Truthfulness

As we mentioned in Section 2.2.1, our equilibrium notion of choice in this thesis would be that of *dominant strategies*. As such, the default notion of truthfulness for our mechanisms is that of DSIC (see Definition 2.4). We do that deliberately, due to the robustness and transparency of such solutions. However, at the same time we do understand that the standard approach of auction design in the economics literature is to a priori focus on the *weaker* notion of BIC truthfulness (see e.g. [47]) instead. Thus we want to let the reader know that, exactly because of the *stronger* solution concept we have chosen, our duality framework can be readily adapted to Bayesian truthfulness with minimal effort: just redefine the utility functions of the players to correspond to their *expected* utility under their prior knowledge of the types of the other players:

$$U_i(\mathbf{x}_i) \equiv \mathbb{E}_{\mathbf{x}_{-i} \sim F_{-i}} \left[u_i(\mathbf{x}_i, \mathbf{x}_{-i}) \right]. \tag{3.8}$$

Notice how now the utilities U_i are defined over the more restricted, player-specific space D_i instead of the entire domain D. The exposition and all proofs in this chapter can then be carried out almost verbatim utilizing (3.8). Finally, we must mention that this discussion is relevant *only in multi-bidder settings*: the definitions of DSIC and BIC (see (2.2) and (2.3)) coincide when just a single buyer is involved.

3.3.4 No-Positive Transfers

In addition to the standard assumptions of truthfulness (IC) and voluntary participation (IR) we presented in Section 2.2.3 there is another common, natural condition known as *No-Positive Transfers (NPT)*:

$$p_i(\mathbf{x}) \geq 0 \qquad \text{for all players } \; i \in [n] \; \text{ and } \; \mathbf{x} \in D.$$

This expresses the desirable property that our mechanisms should only *receive* payments *from* the players and never give positive monetary transfers *back to* them. It can be shown to be equivalent, at least when DSIC truthfulness is used or BIC with product distributional priors (that is, independent items), to the condition

$$u_i(\mathbf{0}, \mathbf{x}_{-i}) = 0 \tag{3.9}$$

which says that, every player who has zero desire for the items should actually get exactly zero utility from the outcome of the auction[5]. The reason that we didn't explicitly include this property in the initial formulation of the revenue maximization problem and thus in the constraints of the primal Program (3.1), is that it can be shown that it comes for free along with the optimal solution; that is, for any auction for which (3.9) does not hold, there is another auction with at least as good revenue that does satisfy this exact equality. For a more detailed exposition on this we refer to Hart and Nisan [40], but the important point the reader needs to keep from this discussion is that, in the rest of our work, *whenever we look into designing good revenue-maximizing auctions, we can without loss focus on these which satisfy condition* (3.9).

[5]Recall that the inequality $u_i(\mathbf{0}, \mathbf{x}_{-i}) \geq 0$ holds anyway due to IR.

Chapter 4

Uniform Distributions

This chapter is dedicated to demonstrating the power and usage of the duality framework developed in Chapter 3, by applying it to the canonical open problem of revenue maximization in the economics literature, that of a single bidder setting where item values come i.i.d. from a uniform distribution over the real unit interval $[0, 1]$. We look for the optimal selling mechanism that a multiple-good monopolist facing a buyer with i.i.d. uniform bids should use in order to maximize his expected revenue. Notice that we do not restrict our attention to just deterministic mechanisms (i.e. price schedules) but we allow for general randomized ones (i.e. lotteries).

4.1 The Primal and Dual Programs

First we make the choice to relax the primal Program (3.1) even further[1] by dropping the nonnegative derivatives constraint; as we shall see, this will end being without loss to optimality. This relaxation translates into dual variables $s_{i,j}$ not appearing in the dual Program (3.3) (see also the discussion in Section 3.2.2). So, to be specific, let us write down how the primal and dual programs become in the current setting of a single buyer ($n = 1$) having i.i.d. uniform bids ($f_j(x_j) = 1$ for all $j \in [m]$). The primal is

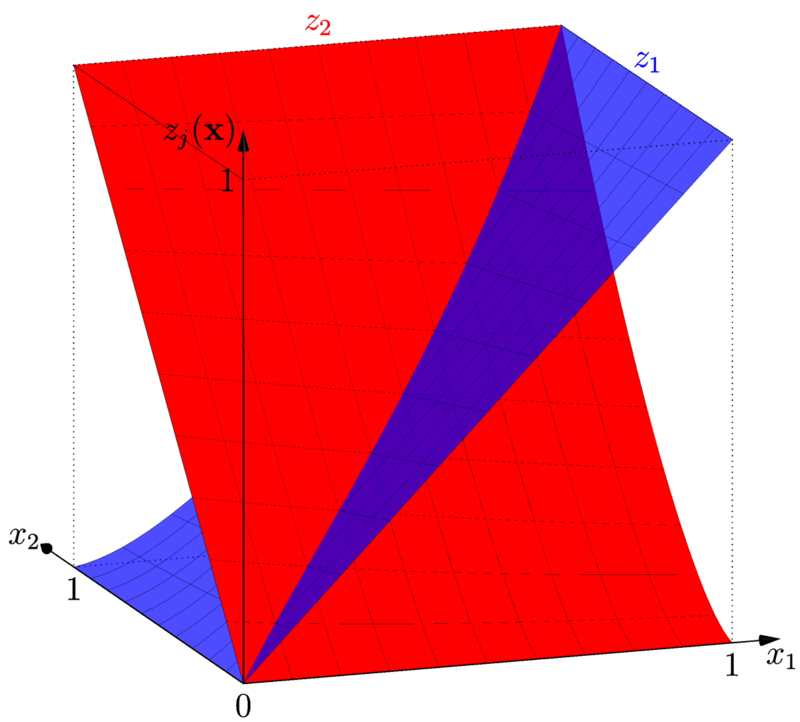
$$\max_u \int_{I^m} \nabla u(\mathbf{x}) \cdot \mathbf{x} - u(\mathbf{x})\, d\mathbf{x} \tag{4.1}$$

over the space of absolutely continuous functions $u : I^m \longrightarrow \mathbb{R}_+$ having derivatives

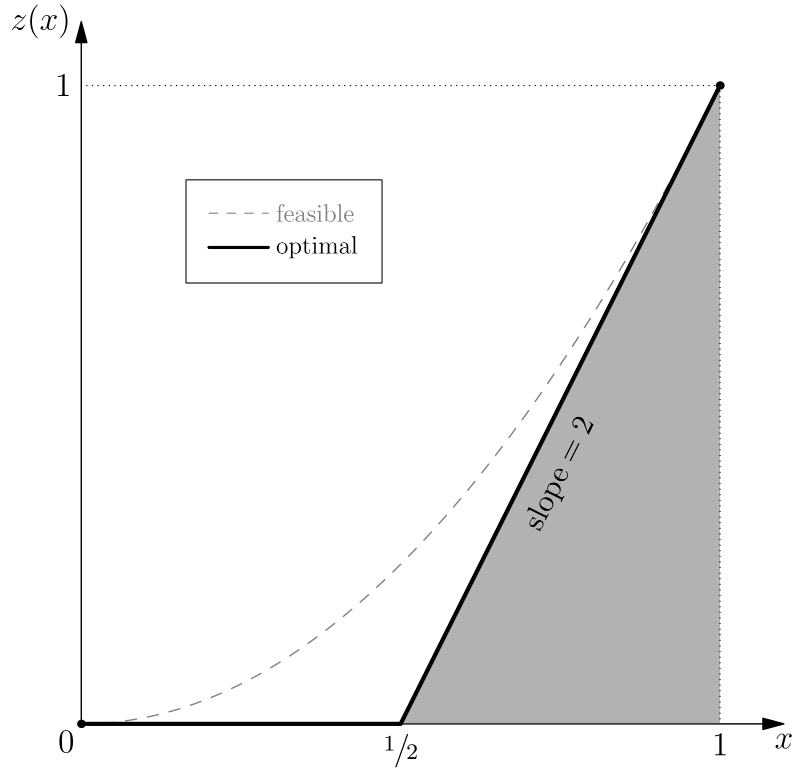
$$\frac{\partial u(\mathbf{x})}{\partial x_j} \leq 1 \qquad (z_j(\mathbf{x}))$$

and the dual

[1] Recall that Program (3.1) is already a relaxed version of the exact revenue maximization problem, since we have dropped the convexity constraint of the utility function.



(a) Feasible solutions z_1, z_2 to the two-items dual program. Each function z_j has to start at 0 on the entire axis $x_j = 0$ and rise to 1. At no point of the 2-dimensional cube the sum of their slopes is allowed to exceed 3, and the objective is to keep them as low as possible, i.e. minimize the volume under their curves.



(b) For the special case of a single item, the dual feasible function z has to start at 0 and rise to 1 or higher when $x = 1$, with a slope of at most 2. The optimal function minimizes the area below it. It is not difficult to see that the optimal solution is to remain at value 0 until $x = 1/2$ and then increase steadily to 1; the optimal dual objective is equal to the grey area. This corresponds exactly to the well-known optimal solution of Myerson with reserve price of 1/2.

Figure 4.1: Geometric interpretation of the dual Program (4.2) for the case of a single-bidder and $m = 1, 2$ uniform i.i.d. items.

$$\min_{z_1,\ldots,z_m} \sum_{j=1}^{m} \int_{I^m} z_j(\mathbf{x})\, d\mathbf{x} \tag{4.2}$$

over the space of absolutely continuous functions $z_j : I^m \longrightarrow \mathbb{R}_+$ with

$$\sum_{j=1}^{m} \frac{\partial z_j(\mathbf{x})}{\partial x_j} \leq m+1 \qquad (u(\mathbf{x}))$$

$$z_j(0, \mathbf{x}_{-j}) = 0 \qquad (u(0, \mathbf{x}_{-j}))$$

$$z_j(1, \mathbf{x}_{-j}) \geq 1. \qquad (u(1, \mathbf{x}_{-j}))$$

A geometric interpretation of this dual for the case of one and two items, based also on the discussion about the duality framework of Section 3.1.2, can be found in Figure 4.1.

4.2 The Straight Jacket Auction (SJA)

The duality conditions are not only useful in establishing optimality, but they can in fact *suggest the optimal auction in a natural way.* We illustrate this by considering the case of two items. Starting from Figure 4.1a, we need to find two functions z_1 and z_2 that satisfy the boundary constraints and the slope constraint. If we had only one function, say z_1, the solution would be obvious and similar to the solution for one item (Figure 4.1b): $z_1(\mathbf{x})$ would be 0 up to $x_1 = 2/3$ and then increase with a maximum slope of 3. But if we do the same for both functions z_1 and z_2, we obtain an infeasible solution: in the square $[2/3, 1] \times [2/3, 1]$ the total slope would be 6 instead of 3. This implies that the functions need more space to grow; in fact, the area of growth needs to be at least equal to the area of the square $[2/3, 1] \times [2/3, 1]$. The natural way to get this space is to add a triangle of area $1/9$ in the way indicated in the left part of Figure 4.2 (the triangle defined by the lines $x_j = p_1 = 2/3$, $j = 1, 2$, and $x_1 + x_2 = p_2$). We then seek a dual solution in which only z_j grows in area $U_{\{j\}}$ and both functions grow in $U_{\{1,2\}}$ (Figure 4.2). The corresponding primal solution is that only item j is sold in $U_{\{j\}}$ and both items are sold in $U_{\{1,2\}}$.

The remarkable fact is that the optimal mechanism is completely determined by the obvious requirement that the area of the triangle must be (at least) equal to $1/9$. To put it in another way: suppose that we knew that the optimal mechanism is deterministic; then the dual program requires that

- the price p_1 for one item must satisfy $p_1 \leq 2/3$ so that z_1 has enough space to grow from 0 to 1 with the maximum slope 3
- the price p_2 for the bundle of both items must be such that the area of the region $\overline{U}_\emptyset = U_{\{1\}} \cup U_{\{2\}} \cup U_{\{1,2\}}$, in which the mechanism allocates at least one item, is at least $2/3$ so that both functions have enough space to grow up to 1

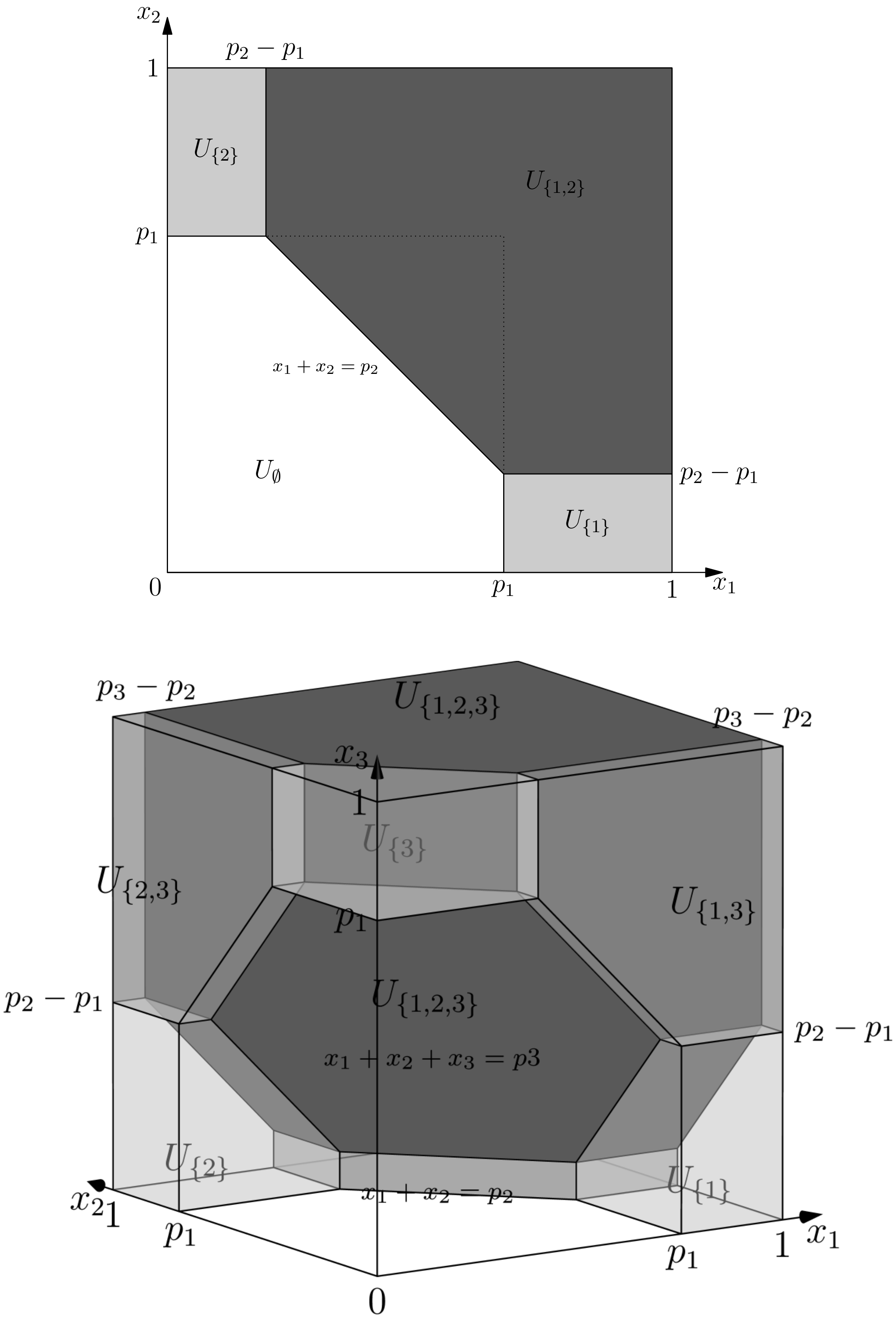


Figure 4.2: The allocation spaces of the optimal SJA mechanisms for $m = 2$ and $m = 3$ items. The payments are given by $p_1 = \frac{m}{m+1}$, $p_2 = \frac{2m-\sqrt{2}}{m+1}$, and $p_3 = 3 - \frac{7.0971}{m+1}$. The mechanism sells at least one item within the grey areas $\overline{U}_\emptyset$, and all items within the *dark* grey areas $U_{[m]}$. If we flip around these dark grey areas by $\mathbf{x} \mapsto \mathbf{1} - \mathbf{x}$, so that $\mathbf{1}$ is mapped to the origin $\mathbf{0}$, they are exactly the SIM-bodies defined in Section 4.3.2, for $k = \frac{1}{m+1}$. These SIM-bodies can be seen in Figures 4.3a and 4.3b, respectively.

The central point of our work is that these necessary conditions (which we call *slice conditions*) are also sufficient. This intuition naturally extends to more items: the price for a bundle of r items is determined by the slice condition that the r-dimensional volume in which the mechanism sells at least one item of the bundle is exactly equal to $r/(m+1)$.

Using this intuition, we define here the *Straight-Jacket Auction (SJA)*. This selling mechanism is deterministic and symmetric; as such, it is defined by a payment vector $p^{(m)} = (p_1^{(m)}, \ldots, p_m^{(m)})$; $p_r^{(m)}$ is the price offered by the mechanism to the bidder for every subset of r items, $r \in [m]$. We will drop the superscript when there is no confusion about the number of available items. The utility of the bidder is then given by

$$u(x) = \max_{J \subseteq [m]} \left(\sum_{j \in J} x_j - p_{|J|} \right).$$

The prices are defined by the slice conditions. For a subset of items $J \subseteq [m]$, let $\Pr(J, \mathbf{x}_{-J})$ be the probability that *at least one* item in J is sold when the remaining items have values $\mathbf{x}_{-J}$. The r-th dimensional slice condition is that for every J with $|J| = r$ and every $\mathbf{x}_{-J}$,

$$\Pr(J, \mathbf{x}_{-J}) \geq \frac{|J|}{m+1}.$$

The SJA is the deterministic mechanism which satisfies the slice conditions for all dimensions *as tightly as possible* (hence its name), in the following sense: determine the prices $p_1, p_2, \ldots, p_m$ in this order so that, having fixed the previous ones, select p_r as large as possible to satisfy all r-dimensional slice conditions. In particular, this guarantees that the m-dimensional slice is tight, or equivalently, that the probability that at least one item is sold is $\frac{m}{m+1}$.

Definition 4.1 (Straight-Jacket Auction (SJA))**.** SJA for m items is the deterministic symmetric selling mechanism whose prices $p_1^{(m)}, \ldots, p_m^{(m)}$, where $p_r^{(m)}$ is the price of selling a bundle of size r, are determined as follows: for each $r \in [m]$, having fixed $p_1^{(m)}, \ldots, p_{r-1}^{(m)}$, price $p_r^{(m)}$ is selected to satisfy

$$\Pr_{\mathbf{x} \sim \mathcal{U}^m} \left[\bigwedge_{J \subseteq [r]} \sum_{j \in J} x_j < p_{|J|}^{(m)} \right] = 1 - r \cdot k \tag{4.3}$$

where $k = \frac{1}{m+1}$. In words, $p_r^{(m)}$ is selected so that the probability of selling no item when r values are drawn from the uniform probability distribution (and the remaining values of the $m - r$ items are set to 0) is equal to $1 - r \cdot k$. We will refer to constraints (4.3) as *slice conditions*.

If we take the complement of the above probability, an equivalent definition would be to ask for the probability of selling at least one of items $[r]$, when all other bids for items $[r+1...m]$ are fixed to zero, to be rk. That is, if for any dimension m and positive $\alpha_1, \alpha_2, \ldots, \alpha_m$ we define

$$V(\alpha_1, \ldots, \alpha_m) \equiv \left\{ \mathbf{x} \in I^m \;\middle|\; \bigvee_{J \subseteq [m]} \sum_{j \in J} x_j \geq \alpha_{|J|} \right\}, \tag{4.4}$$

the volume of the r-dimensional body $V(p_1^{(m)}, \ldots, p_r^{(m)})$, let us denote it by $v(p_1^{(m)}, \ldots, p_r^{(m)})$, must be rk (for all $r \in [m]$).

The specific value on the right-hand side of (4.3) depends on the parameter k, which, in turn, depends on the total number of items m; the exact dependence arises from the specific values of the primal and dual program. It is however useful in providing a unifying approach to carry out the discussion and analysis for an arbitrary (albeit small, $k \leq \frac{1}{m+1}$) parameter k and to plug in the specific value $k = 1/(m+1)$ only when this is absolutely necessary.

It is not immediate that SJA is well-defined. In order for the mechanism to be well-defined, there should be prices $p_r^{(m)}$ that satisfy (4.3). For $m \leq 6$, the main technical result of this work is showing that the mechanism is both well-defined and optimal:

Theorem 4.1. *The Straight-Jacket Auction mechanism is well-defined and optimal for uniform i.i.d. valuations, for up to 6 items.*

Our proof of this theorem relies significantly on the geometry of these mechanisms. We conjecture that the theorem holds for any number of items:

Conjecture. *The Straight-Jacket Auction mechanism is well-defined and optimal for uniform i.i.d. valuations and any number of items.*

Here is how to use the slice conditions (4.3) to compute the prices of SJA: The 1-dimensional condition on a 1-dimensional hypercube simply means that $p_1^{(m)} = 1 - 1/(m+1)$, because we only have condition $x_1 < p_1^{(m)}$. The 2-dimensional condition on a 2-dimensional boundary requires that the region

$$\{\mathbf{x} \mid x_1 + x_2 < p_2 \textbf{ and } x_1 < p_1 \textbf{ and } x_2 < p_1\}$$

inside the unit square must have area equal to $1 - 2/(m+1)$. In other words, we want to find where to move the line $x_1 + x_2 = p_2$ so that the area that it cuts satisfies the slice condition (in the left Figure 4.2, $U_{\{1\}}$, $U_{\{2\}}$, and $U_{\{1,2\}}$ have total volume $\frac{2}{m+1}$); this gives $p_2 = 2 - \frac{2+\sqrt{2}}{m+1}$. We can proceed in the same way to higher dimensions: fix some dimension m and an order $r > 1$. If the prices $p_1, p_2, \ldots, p_r$ are such that $p_j - p_{j-1}$

is a nonnegative and (weakly) decreasing sequence, then

$$v(p_1,\dots,p_r) = \int_0^{p_r-p_{r-1}} v(p_1,\dots,p_{r-1})\,dt + \int_{p_r-p_{r-1}}^{p_{r-1}-p_{r-2}} v(p_1,\dots,p_{r-2},p_r-t)\,dt$$
$$+\dots+\int_{p_2-p_1}^{p_1} v(p_2-t,\dots,p_{r-1}-t,p_r-t)\,dt + \int_{p_1}^{1} 1\,dt \tag{4.5}$$

This is a recursive way to compute the expressions for the volumes $v(p_1,\dots,p_r)$. In case that the sequence $p_1,p_2,\dots,p_r$ of the prices up to order r breaks the requirement to be increasing at the last step, i.e. $p_r < p_{r-1}$, then simply $v(p_1,\dots,p_r) = v(p_1,\dots,p_{r-2},p_r,p_r)$ and we can still deploy the previous recursion.

An exact, analytic computation of these values for up to $r=6$ using the above recursion is given in [32], but we also list them below for quick reference. In the following we will often use the transformation

$$p_r = r - \frac{\mu_r}{m+1} \tag{4.6}$$

so that prices will be determined with respect to some parameters μ_r. It will be algebraically convenient to also assume $p_0 = 0$.

- For $r \le 4$ and *any* number of items $m \ge r$:

$$p_1 = \frac{m}{m+1} \qquad p_2 = \frac{2m-\sqrt{2}}{m+1} \qquad p_3 \approx 3 - \frac{7.0972}{m+1} \qquad p_4 \approx 4 - \frac{11.9972}{m+1}$$
$$\mu_1 = 1 \qquad \mu_2 = 2+\sqrt{2} \qquad \mu_3 \approx 7.0972 \qquad \mu_4 \approx 11.9972$$

- For $r = 5, 6$:

$$p_5^{(5)} \approx 5 - \frac{18.0865}{6} \qquad p_5 \approx 5 - \frac{18.0843}{m+1}\ \ (m \ge 6) \qquad p_6^{(6)} \approx 6 - \frac{25.3585}{7}$$
$$\mu_5^{(5)} \approx 18.0865 \qquad \mu_5 \approx 18.0843\ \ (m \ge 6) \qquad \mu_6^{(6)} \approx 25.3585$$

4.3 Geometric Properties

4.3.1 Bodies and Deficiency

In this section we develop the geometric theory that captures the critical structural properties of SJA mechanisms and use this to prove our main result, Theorem 4.1, that shows their optimality. First we will need to establish some notation and formally define some notions.

For any positive integer m, an m-dimensional *body* A is any compact subset of the nonnegative orthant $A \subseteq \mathbb{R}_+^m$. We will denote its volume simply by $|A| \equiv \mu(A)$ (where μ is the standard m-dimensional Lebesgue measure). For any index set $J \subseteq [m]$, the *projection* of A with respect to the J coordinates is defined as

$$A_{[m]\setminus J} \equiv \{\mathbf{x}_{-J} \mid \mathbf{x} \in A\}$$

and is the remaining body of A if we "delete" coordinates J. For any $r \in [m]$, index set $J \subseteq [m]$ with $|J| = m - r$ and $\mathbf{t} \in \mathbb{R}_+^{m-r}$ we define the *slice* of A at the point $\mathbf{t}$ with respect to coordinates J as

$$A|_{J:\mathbf{t}} \equiv \{\mathbf{x}_{-J} \mid \mathbf{x} \in A \;\wedge\; \mathbf{x}_J = \mathbf{t}\}\,.$$

It is the remaining of the body A if we fix a vector $\mathbf{t}$ at coordinates J. The operations of projecting and slicing bodies commute with each other, that is $A_{[m]\setminus I}\Big|_{J:\mathbf{t}} = (A|_{J:\mathbf{t}})_{[m]\setminus I}$ for all disjoint sets of indices $I, J \subseteq [m]$ and $|J|$-dimensional vector $\mathbf{t}$.

For any set of points $S \subseteq \mathbb{R}_+^m$ we denote their convex hull by $\mathcal{H}(S)$ and for any vector $\mathbf{x}$ we will denote by $\mathcal{P}(\mathbf{x})$ the set of all permutations of $\mathbf{x}$. We will say that a body A is *downwards closed* if for any point of A, all points below it are also contained in A: $\mathbf{y} \in A$ for all $\mathbf{y} \in \mathbb{R}_+^m$ with $\mathbf{y} \leq \mathbf{x} \in A$. Body A will be called *symmetric* if it contains all permutations of its elements: $\mathcal{P}(\mathbf{x}) \subseteq A$ for all $\mathbf{x} \in A$. If an m-dimensional body A is symmetric then one can define its *width* to be the length of its projection towards any axis:

$$w(A) \equiv |A_{\{j\}}| \quad \text{for any } \; j \in [m].$$

In a similar way, if $A \subseteq S$ we will say that A is *upwards closed* (with respect to S) if for any $\mathbf{x} \in A$, we have $\mathbf{y} \in A$ for any $\mathbf{x} \leq \mathbf{y} \in S$. For any set of points $S \subseteq \mathbb{R}_+^m$, its *downwards closure* is defined to be all points below it:

$$\mathcal{D}(S) = \left\{\mathbf{x} \in \mathbb{R}_+^m \mid \exists \mathbf{y} \in S : \mathbf{x} \leq \mathbf{y}\right\}.$$

Finally, we describe a property that will play a key role in the following:

Definition 4.2 (p-closure)**.** We will say that a body A is *p-closed* if it contains the convex hull of the permutations of any of its elements. Formally:

$$\mathcal{H}(\mathcal{P}(\mathbf{x})) \subseteq A \qquad \text{for all } \; \mathbf{x} \in A.$$

Notice that any p-closed body must be symmetric (but not necessarily convex) and that any convex symmetric body is p-closed.

We next define the notion of deficiency of a body, which is one of the key geometric ingredients of our results in this chapter:

Definition 4.3 (Deficiency)**.** For any $k > 0$, we will call *k-deficiency* of a body $A \subseteq \mathbb{R}_+^m$ the quantity

$$\delta_k(A) \equiv |A| - k\sum_{j=1}^{m}\left|A_{[m]\setminus\{j\}}\right|. \tag{4.7}$$

This is inspired by the deficiency notion in bipartite graphs defined by Ore [64]. Here we extend it to the continuous settings, trying to capture how large an m-dimensional body A is with respect to its $(m-1)$-dimensional projections (formal definitions are given in Section 4.3.1). We will sometimes drop the subscript k in the notation above, and simply refer to "deficiency", if the value of parameter k is made clear by the context or is currently irrelevant.

A useful, trivial to prove property of the deficiency function (see Definition 4.3) is that it is *supermodular*:

Lemma 4.1. *For any bodies A_1, A_2,*

$$\delta(A_1 \cup A_2) + \delta(A_1 \cap A_2) \geq \delta(A_1) + \delta(A_2).$$

The next lemma tells us that "leaving gaps" between the points of bodies and the orthant's faces can only reduce the deficiency.

Lemma 4.2. *For any bodies A, B such that $B \subseteq A$ and A being downwards closed, there exists a downwards closed sub-body $\tilde{B} \subseteq A$ such that $\delta(\tilde{B}) \geq \delta(B)$.*

Instead of proving this lemma, we provide a stronger construction, given by the following Lemma 4.3.

Lemma 4.3. *Let $\mathcal{A}_m$ be the set of m-dimensional bodies and $\mathcal{K}_m \subseteq \mathcal{A}_m$ be the set of downwards closed ones. There is a mapping $\chi : \mathcal{A}_m \to \mathcal{K}_m$ such that for every m-dimensional bodies A and B:*

1. *$|\chi(A)| = |A|$ and for every $J \subseteq [m]$, $|\chi(A)_J| \leq |A_J|$.*
2. *$\chi(A) \cup \chi(B) \subseteq \chi(A \cup B)$. Equivalently, $A \subseteq B$ implies $\chi(A) \subseteq \chi(B)$.*
3. *if $A \in \mathcal{K}_m$ then $\chi(A) = A$.*

It is straightforward to see how Lemma 4.3 implies Lemma 4.2, by taking $\tilde{B} = \chi(B)$. Then, $\tilde{B}$ has the same volume as B and (weakly) smaller projections (Property 1). This directly implies that $\delta(\tilde{B}) \geq \delta(B)$. It is also a subset of A (by Property 2): $\tilde{B} = \chi(B) \subseteq \chi(A) = A$; the last equality follows from the fact that A is already downwards closed and thus invariant under χ (Property 3).

Proof of Lemma 4.3. The lemma is proved by induction on m. For $m = 1$ it is trivial: $\chi(A)$ is the interval starting at 0 with length equal to $|A|$.

Fix now a coordinate $j \in [m]$ and consider the $(m-1)$-dimensional slices $A|_{\{j\}:t}$ of A, ranging over t. Apply the lemma recursively (that is, use function χ by the induction hypothesis from the previous dimension) to each such slice to obtain a body A'. Let χ' be this map from $\mathcal{A}_m$ to $\mathcal{A}_m$, i.e. $\chi'(A) = A'$. Notice that A' may not be downwards closed, but we argue that χ' satisfies all three properties.

Indeed, for Property 1, we have two cases: If $j \in J$ then, by using Property 1, we get

$$|A'_J| = \int_t \left|A'_J|_{\{j\}:t}\right| = \int_t \left|\left(A'|_{\{j\}:t}\right)_J\right| = \int_t \left|\left(\chi'(A|_{\{j\}:t})\right)_J\right| \leq \int_t \left|\left(A|_{\{j\}:t}\right)_J\right| = \int_t \left|A_J|_{\{j\}:t}\right| = |A_J|.$$

In particular, the above holds with equality when $J = [m]$. Otherwise, if $j \notin J$, we can deploy Property 2 to get

$$|A'_J| = \left|\left(\bigcup_t A'|_{\{j\}:t}\right)_J\right| = \left|\left(\bigcup_t \chi'\left(A|_{\{j\}:t}\right)\right)_J\right| \leq \left|\left(\chi'\left(\bigcup_t A|_{\{j\}:t}\right)\right)_J\right| \leq \left|\left(\bigcup_t A|_{\{j\}:t}\right)_J\right| = |A_J|.$$

Property 2 is also satisfied because if $A \subseteq B$ then for every t: $A|_{\{j\}:t} \subseteq B|_{\{j\}:t}$, and thus by induction $\chi'(A|_{\{j\}:t}) \subseteq \chi'(B|_{\{j\}:t})$, therefore

$$\mathbf{x} \in \chi'(A) \implies \mathbf{x}_{-j} \in \chi'(A|_{\{j\}:x_j}) \implies \mathbf{x}_{-j} \in \chi'(B|_{\{j\}:x_j}) \implies \mathbf{x} \in \chi'(B).$$

Property 3 is satisfied, since if A is already downwards closed, its slices are also downwards closed and, by induction, they will remain unaffected by χ'.

If A is downwards closed with respect to some coordinate $i \in [m]$, then $\chi'(A)$ will remain closed downwards with respect to i: It is obvious by induction that χ' preserves downwards closure for every coordinate $i \neq j$. For coordinate $i = j$, it suffices to notice that downwards closure of A is equivalent to $A|_{\{j\}:t} \subseteq A|_{\{j\}:t'}$ for all $t \geq t'$. Since χ' satisfies Property 2, the same holds for their images: $\chi'(A|_{\{j\}:t}) \subseteq \chi'(A|_{\{j\}:t'})$.

Map χ' is not the desired map, because if A is not already downwards closed with respect to j, the result may not be downwards closed. However, we can select another coordinate $j' \neq j$ to create another map χ'' similar to χ'. Since χ'' will satisfy all properties and preserve the downwards closure of coordinate j', we conclude that $\chi = \chi'' \circ \chi'$ has all the desired properties. ☐

The supermodularity of deficiency functions (Lemma 4.1) immediately implies that if bodies $A_1, A_2 \subseteq S$ are of maximum deficiency (within S), then both their union and intersection are also of maximum deficiency. Based on this, the following can be shown:

Lemma 4.4. *For any downwards closed and symmetric body A, there is a maximum volume sub-body of A of maximum deficiency, which is also downwards closed and symmetric.*

Proof. Let $B \subseteq A$ be of maximum deficiency. Then, by Lemma 4.2 there exists a downwards closed $\tilde{B} \subseteq A$ such that $\delta(\tilde{B}) \geq \delta(B)$, and due to the maximum deficiency of B, it must be that $\delta(\tilde{B}) = \delta(B)$. Now, let $\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_{m!}$ be all possible permutations of the body $\tilde{B}$ (within the m-dimensional space) and take their union $\hat{B} = \bigcup_{i=1}^{m!} \tilde{B}_i$. This new body $\hat{B}$ is clearly symmetric. Also, because of the symmetry of A, all $\tilde{B}_i$ remain within A, so $\hat{B} \subseteq A$.

Now notice that all B_i's have $\delta(\tilde{B}_i) = \delta(\tilde{B})$, so they also have maximum deficiency within A. Remember that the deficiency function is supermodular (Lemma 4.1), so the union of maximum deficiency sets must also be of maximum deficiency. Thus, $\hat{B}$ is indeed of maximum deficiency. Finally, it is not difficult to see that union preserves downwards closure and also, trivially, $|\hat{B}| \geq |\tilde{B}|$. □

The next lemma describes how global maximum deficiency implies also a kind of local one:

Lemma 4.5. *Let $A \subseteq S$ be a maximum deficiency body (within S). Then,* every *slice of A must have nonnegative deficiency and must not contain subsets with higher deficiency.*

Proof. To get to a contradiction, suppose that there exists such a slice $B = A|_{J:\mathbf{t}}$ of A, such that $\delta(B) < 0$. Then, let us remove the entire slice B above $\mathbf{t}$ from body A, to get a new body A'. This $(m-1)$-dimensional slice though is of measure 0 in the larger m-dimensional space, so what we should really do is to remove an ε-neighbourhood of B (around $\mathbf{t}$) within A, of "parallel" slices. This neighbourhood has a volume of positive measure and is arbitrarily close to the slice[2]. This section removed from the body had the property to have volume strictly less than k times its projections with respect to the coordinates not in J, i.e. the "active" coordinates in B (because we are working close to B for which $\delta(B) < 0$). Regarding the other remaining projections with respect to the coordinates in J, by removing points they cannot possibly be increased. Since volumes have positive sign effect at the expression (4.7) of the deficiency function, and projections have negative, we can deduce that the resulting body has strictly higher deficiency than A, which contradicts the maximum deficiency of A within S.

The proof for subsets of the slice with higher deficiency is similar: replace the entire slice with its subset of higher deficiency, and the total deficiency must increase. □

As a consequence of Lemma 4.5 we get the following properties of maximum deficiency sub-bodies, which imply that these bodies must be "large enough" (Lemmas 4.7 and 4.8) and also demonstrate some kind of "symmetric convexity" (p-closure Lemma 4.9, Definition 4.2). But first we will need an inequality that brings together volumes and projections of bodies, due to Loomis and Whitney [50]. An easy proof of this can be found in [3].

Lemma 4.6 (Loomis-Whitney)**.** *For any m-dimensional body A,*

$$|A|^{m-1} \leq \prod_{j=1}^{m} \left|A_{[m]\setminus\{j\}}\right|.$$

[2]For ease of presentation, in the following we will use that procedure without making explicit mention to the underlying technicalities.

Lemma 4.7. *Let $A \neq \emptyset$ be an m-dimensional body with nonnegative k-deficiency. Then*

$$|A| \geq (km)^m.$$

As a consequence, if A is also symmetric and downwards closed, its width must be at least

$$w(A) \geq km.$$

Proof. Since $\delta_k(A) \geq 0$, we know that

$$|A| \geq k \sum_{j=1}^{m} \left| A_{[m]\setminus\{j\}} \right|,$$

or equivalently

$$\sum_{j=1}^{m} \left| A_{[m]\setminus\{j\}} \right| \leq \frac{|A|}{k}. \tag{4.8}$$

Also, by the Loomis-Whitney inequality (Lemma 4.6):

$$|A|^{m-1} \leq \prod_{j=1}^{m} \left| A_{[m]\setminus\{j\}} \right|.$$

So, by using the arithmetic–geometric means inequality we can derive that

$$|A|^{m-1} \leq \left(\frac{1}{m} \sum_{j=1}^{m} \left| A_{[m]\setminus\{j\}} \right| \right)^m,$$

or equivalently

$$\sum_{j=1}^{m} \left| A_{[m]\setminus\{j\}} \right| \geq m \, |A|^{\frac{m-1}{m}}. \tag{4.9}$$

Combining (4.8) and (4.9) we get

$$m \, |A|^{\frac{m-1}{m}} \leq \frac{|A|}{k},$$

which completes the proof of the lemma (since $|A| \neq 0$). The inequality involving the body's width follows immediately from the observation that every symmetric and downwards closed body A is included in the m-dimensional hypercube with edge length $w(A)$. □

Lemma 4.8. *If A is a nonempty symmetric, downwards closed body with nonnegative deficiency then it must contain the point $(k, 2k, \dots, mk)$. More generally, it must contain the point $(k, 2k, \dots, (m-1)k, w(A))$.*

Proof. We will recursively utilize Lemmas 4.5 and 4.7 to show that points

$$(mk, \mathbf{0}_{m-1}), (mk, (m-1)k, \mathbf{0}_{m-2}), \dots, (mk, (m-1)k, \dots, k)$$

belong to $\hat{A}$, where $\hat{A}$ is a symmetric, downwards closed sub-body of A of maximum deficiency (see Lemma 4.4). By Lemma 4.7 it must be that that $w(\hat{A}) \geq mk$, thus $(mk, \mathbf{0}_{m-1}) \in \hat{A}$ by downwards closure. For the next dimension, consider the slice $\hat{A}\big|_{\{j\}:mk}$ (for some $j \in [m]$). It is $(m-1)$-dimensional, of nonnegative deficiency by Lemma 4.5 and so it must have width at least $(m-1)k$ (Lemma 4.7). That means that point $(mk, (m-1)k, \mathbf{0}_{m-2})$ must be in $\hat{A}$. We can continue like this all the way down to single-dimensional lines. $\square$

Lemma 4.9 (p-closure)**.** *Let $A \subseteq S$ be a maximum volume sub-body of S of maximum deficiency and let S be p-closed. Then* every *slice of A (including A itself) must be p-closed (see Definition 4.2).*

Proof. Without loss (by Lemma 4.4) A can be assumed to be symmetric and downwards closed. We need to prove that for any $r \in [m]$ (r is the dimension of the slice) and any r-dimensional vector $\mathbf{x}$ and $\mathbf{z} \in \mathcal{H}(\mathcal{P}(\mathbf{x}))$ in the convex hull of its permutations:

$$\text{for all } \mathbf{t}: \qquad (\mathbf{x}, \mathbf{t}) \in A \quad \Longrightarrow \quad (\mathbf{z}, \mathbf{t}) \in A.$$

The proof is by induction on r. For the base case of $r = 1$, it is $\mathcal{H}(\mathcal{P}(\mathbf{x})) = \{\mathbf{x}\}$ so the proposition follows trivially. For the induction step, assume the proposition is true for some $r \leq m-1$ and we will prove it for $r+1$. So, take $(r+1)$-dimensional vectors $\mathbf{x}$ and $\mathbf{z}$ such that $\mathbf{z} \in \mathcal{H}(\mathcal{P}(\mathbf{x}))$ and fix some $\mathbf{t} \in \mathbb{R}_+^{m-r-1}$. To complete the proof we need to show that the slice of A above $\mathbf{x}$, with respect to the first $r+1$ coordinates, is included within the one above $\mathbf{z}$, i.e. $A\big|_{[r+1]:\mathbf{x}} \subseteq A\big|_{[r+1]:\mathbf{z}}$. For simplicity, lets abuse notation for the remaining of this proof and just use $A_\mathbf{x}$ and $A_\mathbf{z}$ for these slices.

So, to arrive at a contradiction, let us assume that $A_\mathbf{x} \setminus A_\mathbf{z} \neq \emptyset$. First notice that since $A_\mathbf{x} \cap A_\mathbf{z} \subseteq A_\mathbf{x}$ and $A_\mathbf{x}$ is a slice of a maximum deficiency body, by Lemma 4.5 it must be that $\delta(A_\mathbf{x} \cap A_\mathbf{z}) \leq \delta(A_\mathbf{x})$. So, by the supermodularity of deficiencies (Lemma 4.1) we get that

$$\delta(A_\mathbf{x} \cup A_\mathbf{z}) \geq \delta(A_\mathbf{z}).$$

This means that if we replace (an ε-neighbourhood around $\mathbf{z}$ of) slice $A_\mathbf{z}$ by its superset $A_\mathbf{x} \cup A_\mathbf{z}$ and we can also show that no new projections are created with respect to the first $r+1$ coordinates, then the overall deficiency of the body would not decrease and its volume would increase strictly (since we have assumed that $A_\mathbf{x} \setminus A_\mathbf{z} \neq \emptyset$), which is a contradiction to the maximum deficiency of A within S. Notice a subtle point here: how do we know that this extension can fit within S above point $\mathbf{z}$? It does, because we have assumed S to be p-closed and the new elements added are convex combinations of permutations of elements already known to be in S. The remaining of the proof is dedicated to proving that this extension indeed does not create new projections with respect to the first $r+1$ coordinates.

Without loss, due to symmetry, we can take $x_1 \leq x_2 \leq \cdots \leq x_{r+1}$. We argue that, if we remove any one of the coordinates of the vector $\mathbf{z}$, it can be dominated by a convex combination of permutations of the vector $\mathbf{x}_{-1}$ (i.e. the vector $\mathbf{x}$ if we remove its *smallest* coordinate). To see that, remember that $\mathbf{z}$ is at the convex hull of the permutations of $\mathbf{x}$, so there exist nonnegative real parameters $\{\xi_\pi\}$ such that

$$\mathbf{z} = \sum_{\pi \in \mathcal{P}(\mathbf{x})} \xi_\pi \pi \quad \text{and} \quad \sum_{\pi \in \mathcal{P}(\mathbf{x})} \xi_\pi = 1.$$

But that means that

$$\mathbf{z}_{-j} = \sum_{\pi \in \mathcal{P}(\mathbf{x})} \xi_\pi \pi_{-j} \tag{4.10}$$

for any coordinate j. Now let us define a transformation ϕ over all vectors

$$\{\pi_{-j} \mid \pi \in \mathcal{P}(\mathbf{x}) \;\text{ and }\; j \in [r+1]\}$$

such that $\phi(\pi_{-j}) = \pi_{-j}$ if the j-th coordinate removed from π to get π_{-j} was x_1, and otherwise $\phi(\pi_{-j})$ is the r-dimensional vector that we get if we replace x_1 in π_{-j} by the coordinate π_j that was removed. It follows that for all j

$$\pi_{-j} \leq \phi(\pi_{-j}) \quad \text{and} \quad \phi(\pi_{-j}) \in \mathcal{P}(\mathbf{x}_{-1}),$$

so by (4.10):

$$\mathbf{z}_{-j} \leq \sum_{\pi \in \mathcal{P}(\mathbf{x})} \xi_\pi \phi(\pi_{-j}) \in \mathcal{H}(\mathcal{P}(\mathbf{x}_{-1})).$$

By the induction hypothesis and downwards closure for A it can be deduced that

$$(\mathbf{x}_{-1}, 0, \mathbf{t}) \in A \implies (\mathbf{z}_{-j}, 0, \mathbf{t}) \in A \quad \text{for all } \; j \in [r+1].$$

Thus in particular for every $\mathbf{t} \in A_{\mathbf{x}}$, due to symmetry of A, we have that

$$((\mathbf{z}_{-j}, 0), \mathbf{t}) \in A,$$

which means that indeed every projection of $(\mathbf{z}, \mathbf{t})$ with respect to a coordinate in $[r+1]$ was already included in A. □

4.3.2 SIM Bodies

Definition 4.4 (SIM-bodies)**.** For positive $\alpha_1 \leq \cdots \leq \alpha_r$, let

$$\Lambda(\alpha_1, \ldots, \alpha_r) \equiv \left\{ \mathbf{x} \in \mathbb{R}_+^r \;\middle|\; \bigwedge_{J \subseteq [r]} \left(\sum_{j \in J} x_j \;\leq \sum_{j=r-|J|+1}^{r} \alpha_j \right) \right\}. \tag{4.11}$$

We call these *SIM-bodies*[3]. We will also use the following notation

$$q \cdot \Lambda(\alpha_1, \ldots, \alpha_r) \equiv \Lambda(q \cdot \alpha_1, \ldots, q \cdot \alpha_r)$$

for any positive real q.

The geometry of the allocation space of the SJA mechanisms (see Figure 4.2) naturally gives rise to this family of bodies. Their importance and connection with the structure of the SJA mechanisms will become evident in Section 4.4 where we prove Lemma 4.16. The intuition behind the naming becomes obvious if one looks at Figure 4.3a. By the way SIM-bodies are defined, one can immediately see that they are downwards closed, symmetric and convex polytopes. Thus, they are also p-closed. Each one of its faces corresponds to a defining hyperplane

$$\sum_{j \in J} x_j = \alpha_{r+1-|J|} + \cdots + \alpha_r$$

for some $J \subseteq [r]$ or, of course, to a side of the r-dimensional positive orthant $\mathbb{R}^m_+$.

SIM-bodies demonstrate some inherently recursive and symmetric properties, captured by the following lemma. They are made clear in Figure 4.3.

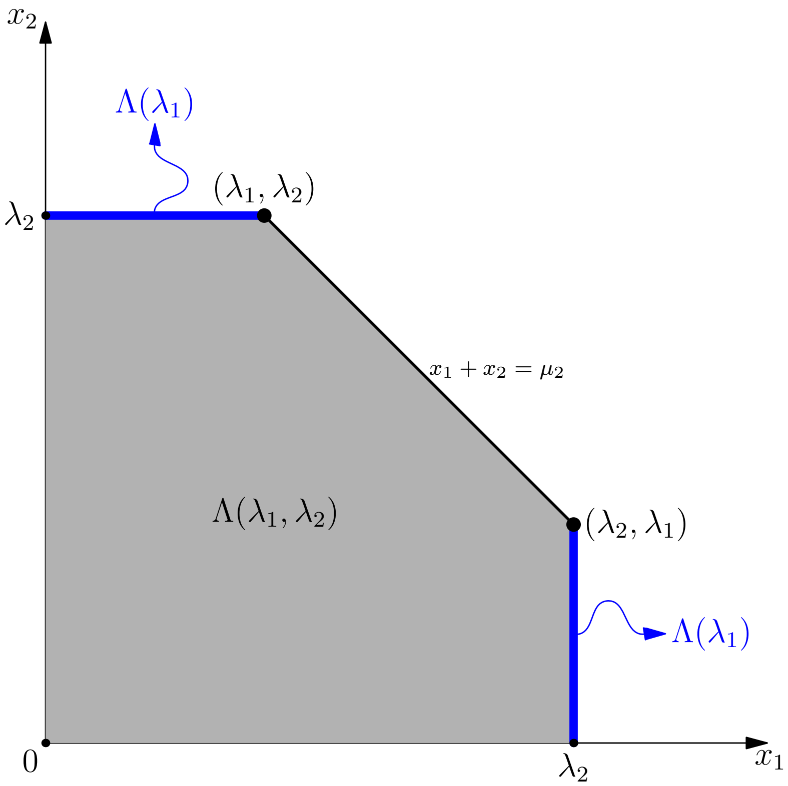
Lemma 4.10. *For any SIM-body $\Lambda = \Lambda(\alpha_1, \ldots, \alpha_r)$:*

1. $w(\Lambda) = \alpha_r$
2. $\Lambda = \mathcal{D}(\mathcal{H}(\mathcal{P}(\alpha_1, \ldots, \alpha_r)))$
3. $\Lambda|_{\{j\}:\alpha_r} = \Lambda(\alpha_1, \ldots, \alpha_{r-1})$ *for any* $j \in [r]$
4. $\Lambda_{[r]\setminus\{j\}} = \Lambda(\alpha_2, \ldots, \alpha_r)$ *for any* $j \in [r]$
5. $\delta_{q \cdot k}(q \cdot \Lambda) = q^r \cdot \delta_k(\Lambda)$ *for any* $q, k > 0$

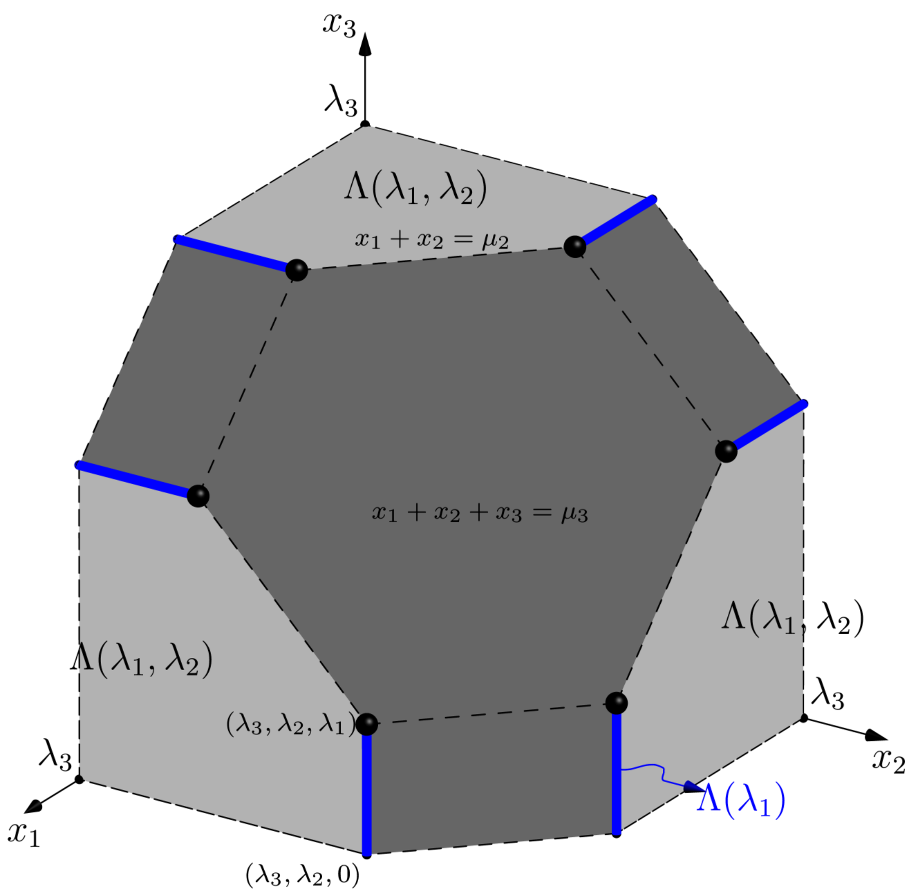
Proof. Property 1 is trivial: by the definition of SIM-bodies (4.11), a point $(x, \mathbf{0}_{r-1}) \in \Lambda$ if and only if $x \leq \alpha_r \ \wedge \ \ldots \ \wedge \ x \leq \alpha_1 + \cdots + \alpha_r$, i.e. $x \leq \alpha_r$.

For Property 2, let E be the set of the extreme points of the polytope Λ. It is convex, thus $\Lambda = \mathcal{H}(E)$. But it is also downwards closed, so we can just focus to the extreme points $\overline{E} \subseteq E$ that belong to the "full" facet of the hyperplane $x_1 + \ldots x_r = p_r$, since the entire polytope can be recovered as the downwards closure $\Lambda = \mathcal{D}(\mathcal{H}(\overline{E}))$. By taking intersections with the other hyperplanes and keeping in mind that the α_j's are non-decreasing, we get that these extreme points in $\overline{E}$ are $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ and all its permutations. So, we can recover the entire SIM-body as $\Lambda = \mathcal{D}(\mathcal{H}(\mathcal{P}(\alpha_1, \ldots, \alpha_r)))$.

[3] The naming is inspired by the shape of the subscriber identity module (SIM) integrated circuit cards used in mobile phone devices (see Figure 4.3a).



(a) The 2-dimensional SIM-body $\Lambda(\lambda_1, \lambda_2)$



(b) The 3-dimensional SIM-body $\Lambda(\lambda_1, \lambda_2, \lambda_3)$

Figure 4.3: SIM-bodies for dimensions $m = 2, 3$. Notice the recursive nature of these constructions: a SIM-body encodes in it the SIM-bodies of lower dimensions as extreme slices (Property 3 of Lemma 4.10). In this figure, these 1-dimensional critical bodies are denoted by thick lines (blue in the colour version) and the 2-dimensional ones in light grey.

For Property 3, notice that an $(r-1)$-dimensional vector $\mathbf{x}$ belongs in the slice $\Lambda|_{\{j\}:\alpha_r}$ if and only if $(\mathbf{x}, \alpha_r) \in \Lambda$, which by using (4.11) is equivalent to

$$\bigwedge_{J \subseteq [r-1]} \left(\sum_{i \in J} x_i \leq \alpha_{r+1-|J|} + \cdots + \alpha_r \right) \quad \text{and} \quad \bigwedge_{J \subseteq [r-1]} \left(a_r + \sum_{i \in J} x_i \leq \alpha_{r-|J|} + \cdots + \alpha_r \right).$$

The second set of conditions can be rewritten simply as

$$\bigwedge_{J \subseteq [r-1]} \sum_{i \in J} x_i \leq \alpha_{r-|J|} + \cdots + \alpha_{r-1} \tag{4.12}$$

which makes the first set of constraints redundant since

$$\alpha_{r-|J|} + \cdots + \alpha_{r-1} \leq \alpha_{r+1-|J|} + \cdots + \alpha_r$$

from the monotonicity of the sequence of α_r's. The constraints (4.12) that we are left with, exactly define $\Lambda(\alpha_1, \ldots, \alpha_{r-1})$ (see (4.11)).

Property 4 can be shown in a very similar way: due to downwards closure, any projection $\Lambda_{[r]\setminus\{j\}}$ is just the slice $\Lambda|_{\{j\}:0}$.

Finally, Property 5 is a result of scaling: $q \cdot \Lambda$ and Λ are similar by a scaling factor of q, so the ratio of their volumes is q^r and the ratio of their projections is q^{r-1}. In formula (4.7) that defines deficiencies, the volumes of the projections are also multiplied by the parameter k of the deficiency, resulting to an overall ratio of q^r between the two deficiencies. □

4.3.3 Matchings

The notion of a matching in a bipartite graph will be very useful for our exposition, so here we recall some basic relevant graph-theoretic facts. Let $G = (V, E)$ be an undirected graph with node set V and edge set E. We will use standard notation and for any set of nodes $X \subseteq V$, $N(X)$ will denote its set of *neighbours*, i.e.

$$N(X) = \{ y \in V \mid (x, y) \in E \text{ for some } x \in X \}.$$

A *matching* $M \subseteq E$ on G is a set of edges with pair-wise no common endpoints, i.e.

$$(x, y), (x', y') \in M \implies x \neq x' \wedge y \neq y'.$$

We will say that M *completely* matches a subset of nodes $V' \subseteq V$ if for any $x \in V'$ there is an edge $(x, y) \in M$ for some $y \in V$.

If G is bipartite with node sets X, Y, i.e. $V = X \cup Y$ and $E \subseteq X \times Y$, Hall's Theorem [51] tells us that there exists a matching in G that completely matches X if and only if

$$|S| \leq |N(S)| \quad \text{for all } S \subseteq X. \qquad \text{(Hall's condition)}$$

4.4 Decomposition of SJA

In this section we bring together all the necessary elements needed to prove Lemma 4.16. We study the structure of the allocation space of SJA that reveals an elegant decomposition which demonstrates that the SIM-bodies essentially act as building blocks for SJA.

Definition 4.5. We denote by $U_J^{(m)}$ the subdomain in which SJA allocates exactly the bundle $J \subseteq [m]$ of items:

$$U_J^{(m)} \equiv \left\{ \mathbf{x} \in I^m \;\middle|\; \bigwedge_{L \subseteq [m]} \sum_{j \in J} x_j - p_{|J|}^{(m)} \geq \sum_{j \in L} x_j - p_{|L|}^{(m)} \right\}. \tag{4.13}$$

Let $U_J^{(m)}\Big|_{-J:\mathbf{t}}$ denote the $|J|$-dimensional slice of $U_J^{(m)}$ when we fix the values of the remaining $[m] \setminus J$ items to $\mathbf{t}$:

$$U_J^{(m)}\Big|_{-J:\mathbf{t}} = \{x_J \,:\, (\mathbf{x}_J, \mathbf{t}) \in U_J\}.$$

For example, the slices of $U_{\{1\}}^{(m)}$ are the horizontal (1-dimensional) intervals; when $J = [m]$, $U_J^{(m)}$ has only one slice, itself. Figure 4.2 shows the various subdomains $U_J^{(m)}$ for $m = 2, 3$.

A simple algebraic manipulation of (4.13), using the non-decreasing property of the SJA payments, gives us the following characterization:

Lemma 4.11. *For any subset of items $J \subseteq [m]$,*

$$U_J^{(m)} = \left\{ \mathbf{x} \in I^m \;\middle|\; \bigwedge_{L \subseteq J} \sum_{j \in L} x_j \geq p_{|J|}^{(m)} - p_{|J|-|L|}^{(m)} \bigwedge_{L \subseteq [m] \setminus J} \sum_{j \in L} x_j \leq p_{|J|+|L|}^{(m)} - p_{|J|}^{(m)} \right\}.$$

Notice here that, due to symmetry, every slice $U_J^{(m)}\Big|_{-J:\mathbf{t}}$ with $|J| = r \leq m$ is isomorphic to $U_{[r]}^{(m)}\Big|_{[r+1\ldots m]:\mathbf{t}}$ and so, from the characterization in Lemma 4.11, this slice is *invariant* with respect to the specific value of the ($(m-r)$-dimensional) vector $\mathbf{t}$. In particular, if it's nonempty, then

$$U_J^{(m)}\Big|_{-J:\mathbf{t}} = U_J^{(m)}\Big|_{-J:\mathbf{0}_{m-|J|}}. \tag{4.14}$$

The following lemma essentially gives an alternative definition of SJA, in terms of the deficiencies of its allocation components $U_J^{(m)}$. In particular, it requires every $|J|$-dimensional slice of any subdomain $U_J^{(m)}$ to have zero deficiency:

Lemma 4.12. *Every slice* $U_J^{(m)}\big|_{-J:\mathbf{t}}$ *of SJA has zero* k*-deficiency, where* $k = \frac{1}{m+1}$.

Proof sketch. Fix some dimension m and let $k = \frac{1}{m+1}$. By the definition of SJA (4.3), the domain $\overline{U}_\emptyset$ where at least one item is sold must have volume $\frac{m}{m+1}$: the probability of selling at least an item is $mk = \frac{m}{m+1}$ which corresponds to the volume of this domain because the valuations' space is the unit cube I^m. Every projection $(\overline{U}_\emptyset)_{\{j\}}$ of this body towards any coordinate $j \in [m]$ has volume 1: it is the $(m-1)$-dimensional side of the cube; just set the valuation of item j to $x_j = 1$ and trivially notice that no matter what the remaining valuations $\mathbf{x}_{-j} \in I^{m-1}$ are, at least one item is being sold by SJA, namely item j, since $x_j = 1 \geq p_1$. Bringing the above together, this means that the k-deficiency of $\overline{U}_\emptyset$ is $m/(m+1) - k \cdot m \cdot 1 = 0$.

This valuations subdomain $\overline{U}_\emptyset$ where at least one item is sold, can be decomposed in its various components U_J, where $\emptyset \neq J \subseteq [m]$. Its volume is just the sum of the volumes of these components. Also, its projections (i.e. the sides of the unit cube I^m) can be covered by taking the projection of any such component U_J with respect to its "active" coordinates in J. This tells us that the deficiency of the entire body $\overline{U}_\emptyset$ is essentially reduced to the sum of the deficiencies of its subdomains. But this body has zero k-deficiency, so all its components must also have zero deficiencies (by using an inductive argument).

A complete, formal proof of this characterization can be found in the following subsection: □

Full Proof of Lemma 4.12

Recall the definition of body $V(p_1^{(m)}, \dots, p_r^{(m)})$ in (4.4). By the definition of SJA in (4.3), the volume of this body must be equal to rk. Then, as we discussed in the proof sketch of Lemma 4.12 in Section 4.4, this translates to its deficiency being zero:

$$\delta_{\frac{1}{m+1}}(V(p_1^{(m)}, \dots, p_r^{(m)})) = 0 \quad \text{for all} \;\; r \leq m. \tag{4.15}$$

Before giving the formal proof of Lemma 4.12, we will need the following lemma that shows how the deficiency of any such subdomain of the valuation space is the sum of the deficiencies of its "critical" sub-slices of lower dimensions:

Lemma 4.13. *For any subset of items* $J \subseteq [m]$*, the* k*-deficiency of any slice of the subdomain where at least one of the items in* J *is sold, when all other items' bids are fixed to zero, is the sum of the* k*-deficiencies of all its sub-slices* $(U_L^{(m)}|_{-J:\mathbf{0}})|_{J\setminus L:\mathbf{t}}$*, where* $\emptyset \neq L \subseteq J$ *and* $k = \frac{1}{m+1}$*. Formally,*

$$\delta_k(V(p_1^{(m)}, \dots, p_{|J|}^{(m)})) = \sum_{\emptyset \neq L \subseteq J} \int_{I^{|J|-|L|}} \delta_k\left(\left(U_L^{(m)}\Big|_{-J:\mathbf{0}} \right)\Big|_{J\setminus L:\mathbf{t}} \right) d\mathbf{t}.$$

Proof. Fix some m. For the sake of clarity we will prove the proposition for J having full dimension $J = m$. All the arguments easily carry on to the more general case where $J \subseteq [m]$ if one takes all valuations of items not in J to be 0, i.e. "slicing" $(\quad\cdot\quad)|_{-J:\mathbf{0}}$, since they are valid for any selling mechanism with non-increasing price differences and SJA specifically; essentially, the case of $|J| = m' \leq m$ directly translates to the case of an m'-dimensional mechanism.

So, it is enough to show that

$$|V| = \sum_{\emptyset \neq L \subseteq [m]} \int_{I^{m-|L|}} \left| U_L|_{-L:\mathbf{t}} \right| \, d\mathbf{t} \tag{4.16}$$

$$\left| V_{[m]\setminus\{j\}} \right| = \sum_{\substack{\emptyset \neq L \subseteq [m] \\ j \in L}} \int_{I^{m-|L|}} \left| \left(U_L|_{-L:\mathbf{t}} \right)_{[m]\setminus\{j\}} \right| \, d\mathbf{t} \qquad \text{for all } \; j \in [m], \tag{4.17}$$

where for simplicity we have denoted the space $V(p_1, \dots, p_m)$ where mechanism allocates at least one item with V. Equation (4.16) is a result of the fact that V can be decomposed as $V = \sum_{\emptyset \neq L \subseteq [m]} U_L$ and every allocation subspace U_L is isomorphic to the *disjoint* union of all its slices $U_L|_{-L:\mathbf{t}}$. In a similar way, to prove that (4.17) holds, it is enough to show that for some fixed $j \in [m]$, the projection $V_{[m]\setminus\{j\}}$ can be covered by the union of all the projections of the sub-spaces U_L with respect to coordinate j and that all these projections $(U_L)_{[m]\setminus\{j\}}$ are disjoint almost everywhere, i.e. they can only intersect in a set of measure zero.

For the former, let $\mathbf{x}_{-j} \in V_{[m]\setminus\{j\}}$. Then $(\mathbf{x}_{-j}, 1) \in V$ (by only increasing the components of a valuation profile, items that were sold to the buyer are still going to be sold). So, there is a nonempty set of items $L \subseteq [m]$ such that $(\mathbf{x}_{-j}, 1) \in U_L$ and $j \in L$ (item j is sold since $x_j = 1 \geq p_1$), meaning that indeed $\mathbf{x}_{-j} \in (U_L)_{[m]\setminus\{j\}}$ with $j \in L$. For the latter, consider distinct sets $L, L' \subseteq [m]$ with j belonging to both L and L', and let a valuation profile $\mathbf{x} \in U_L \cap U_{L'}$. Then, by the characterization in Lemma 4.11 it must be that

$$\sum_{l \in L' \setminus L} x_l \geq p_{|L'|} - p_{|L'|-|L' \setminus L|} \quad \text{and} \quad \sum_{l \in L' \setminus L} x_l \leq p_{|L|+|L' \setminus L|} - p_{|L|},$$

the first inequality being from the fact that $\mathbf{x} \in U_{L'}$ and the second from $\mathbf{x} \in U_L$, taking into consideration that $L' \setminus L \subseteq L'$ and $L' \setminus L \not\subseteq L$. As a result, the sum $\sum_{l \in L' \setminus L} x_l$ can range at most over only a single value, namely $p_{|L'|} - p_{|L'|-|L' \setminus L|} = p_{|L|+|L' \setminus L|} - p_{|L|}$ (and only if these two values are of course equal), otherwise by merging these two inequalities together we would have got that

$$p_{|L'|} - p_{|L'|-|L' \setminus L|} < p_{|L|+|L' \setminus L|} - p_{|L|}$$

which contradicts the non-increasing payment differences property, since both differences are between payments that differ at exactly $|L' \setminus L|$ "steps" but $|L| + |L' \setminus L| \geq |L'|$. $\square$

Lemma (Lemma 4.12). *Every slice* $U_J^{(m)}\big|_{-J:\mathbf{t}}$ *of SJA has zero k-deficiency, where* $k = \frac{1}{m+1}$.

Proof. Fix some m and let $k = \frac{1}{m+1}$. We use induction on the cardinality of J. At the base of the induction, $|J| = 1$ and due to symmetry it is enough to prove the proposition for slices of the form $U_{\{1\}}\big|_{[2...m]:\mathbf{t}}$. By (4.14) this is equal to the slice $U_{\{1\}}\big|_{[2...m]:\mathbf{0}_{m-1}}$, which is the single-dimensional interval $[p_1, 1]$, thus having k-deficiency $1 - p_1 - \frac{1}{m+1} \cdot 1 = \frac{m}{m+1} - p_1 = 0$.

For the inductive step, fix some $r \le m$ and assume the proposition holds for all $J \subseteq [m]$ with $|J| \le r-1$. We will show that it is true also for $|J| = r$. Again, due to symmetry, it is enough to prove that the k-deficiency of slice $U_{[r]}\big|_{[r+1...m]:\mathbf{0}_{m-r}}$ is zero (taking into consideration (4.14)). By Lemma 4.13 we have that for the subdomain where at least one of items $[r]$ is sold given that the remaining $[r+1...m]$ bids are fixed to zero is

$$\begin{aligned}
\delta_k(V(p_1,\dots,p_r)) &= \sum_{\emptyset \ne L \subseteq [r]} \int_{I^{r-|L|}} \delta_k\left(\left(U_L\big|_{[r+1...m]:\mathbf{0}}\right)\Big|_{[r]\setminus L:\mathbf{t}}\right) d\mathbf{t} \\
&= \delta_k\left(U_{[r]}\Big|_{[r+1...m]:\mathbf{0}}\right) + \sum_{\substack{\emptyset \ne L \subseteq [r] \\ |L| \le r-1}} \int_{I^{r-|L|}} \delta_k\left(U_L\big|_{[m]\setminus L:(\mathbf{t},\mathbf{0})}\right) d\mathbf{t} \\
&= \delta_k\left(U_{[r]}\Big|_{[r+1...m]:\mathbf{0}}\right),
\end{aligned}$$

by the induction hypothesis. But from the definition of SJA, from (4.15) we have that $\delta_k(V(p_1,\dots,p_r)) = 0$, which concludes the proof. □

Now let us return to the normal flow of our presentation in Section 4.4, about the decomposition of the allocation space of SJA. The way in which the SJA payments are constructed makes them satisfy a kind of "contraction" property:

Lemma 4.14. *The prices of the SJA mechanism have non-increasing differences, i.e.*

$$p_r^{(m)} - p_{r-1}^{(m)} \le p_{r-1}^{(m)} - p_{r-2}^{(m)}$$

for all $r = 2, \dots, m$.

Proof. Fix some dimension m for which the SJA mechanism is well-defined and assume that we have computed prices up to $p_1, p_2, \dots p_{r-1}$ for some $2 \le r \le m$. First we will show that

$$(r-1)p_r \le r p_{r-1}, \tag{4.18}$$

i.e. that the price p_r must be in $[0, \frac{r}{r-1}p_{r-1}]$. We will do that by showing that otherwise this price would be redundant, in the sense that for any $p_r \ge \frac{r}{r-1}p_{r-1}$ the sub-body of I^r defined by

$$\bigwedge_{J \subseteq [r]} \sum_{j \in J} x_j < p_{|J|}$$

and whose volume must be exactly $1-rk$ in Definition 4.1, would remain unchanged and equal to the one defined by

$$\bigwedge_{\substack{J\subseteq[r]\\|J|\leq r-1}} \sum_{j\in J} x_j < p_{|J|} \tag{4.19}$$

In that case, no such p_r can be a solution of (4.3) (SJA is well-defined).

Indeed, the body defined from (4.19) is a downwards closed, symmetric convex polytope and for the newly inserted hyperplane $x_1+\cdots+x_r=p_r$ to have any effect on it, i.e. to have a non-empty intersection with it, it must be that this hyperplane's "symmetric point" $(\frac{p_r}{r},\ldots,\frac{p_r}{r})$ belongs already to the interior of the body in (4.19) (this is due to the symmetry and convexity of the body). So, this point must satisfy the $(r-1)$-dimensional condition $x_1+\cdots+x_{r-1}\leq p_{r-1}$, thus $(r-1)\frac{p_r}{r}\leq p_{r-1}$ which is exactly property (4.18).

To show that $p_r-p_{r-1}\leq p_{r-1}-p_{r-2}$ for all $2\leq r\leq m$, or equivalently $p_r\leq 2p_{r-1}-p_{r-2}$, by (4.18) it is enough to show that $\frac{r}{r-1}p_{r-1}\leq 2p_{r-1}-p_{r-2}$. But this is equivalent to $(r-1)p_{r-2}\leq(r-2)p_{r-1}$ which we know that holds, also from (4.18). □

Normalized payments. By the procedure of defining SJA payments (Definition 4.1), it can be the case that price p_r is smaller than p_{r-1}, i.e. $p_r\in[p_l,p_{l+1}]$ for some $l\leq r-2$. This is perfectly acceptable, and it just means that essentially we render older prices that are above p_r redundant, in the sense that setting $p_j\leftarrow p_r$ for all $j<r$ with $p_j\geq p_r$ would not have an effect on the sub-body $\bigwedge_{J\subseteq[r]}\sum_{j\in J}x_j<p_{|J|}$ of I^r used in the Definition of SJA in (4.3). This because $x_1+\cdots+x_r\leq p_r\implies x_1+\cdots+x_j\leq p_j$ (since $j<r$ and $p_r\leq p_j$), so old conditions $x_1+\cdots+x_j\leq p_j$ have become useless.

Furthermore, by the non-increasing property of the SJA payments (Lemma 4.14), every new payment after r will continue to fall below the previous one. So, at the end the situation will be in the form of

$$p_1\leq\cdots\leq p_l\leq p_m\leq\ldots \tag{4.20}$$

for some $l<m$ and, as we discussed above, there will be absolutely no effect on the mechanism if we update all older payments that have ended up above p_m to "collapse" to p_m, i.e.

$$p_1\leq\cdots\leq p_l\leq p_m=p_{m-1}=p_{m-2}=\cdots=p_{l+1}. \tag{4.21}$$

Rigorously, we redefine

$$p_j^{(m)}\leftarrow p_m^{(m)}\quad\text{for all }\ j\in[m-1]\ \text{ with }\ p_j\geq p_m.$$

While this *normalization* has no effect on the SJA mechanism itself, it makes sure that payments are now given in a *non-decreasing* order, which is an elegant property that will simplify our exposition later on.

An important observation is that this normalization of payments does not break the property of the non-increasing differences of the payments of SJA, i.e. Lemma 4.14 continues to hold: having a look at the transition before and after the normalization process from (4.20) to (4.21) we see that all the differences up to the l-th payment remain unchanged, $p_{l+1} - p_l$ can only decrease and all differences above the $(l+1)$-th payment have just collapsed to 0.

From now on and for the remaining of this chapter we will assume that SJA payments are normalized. The only difference that this makes, for up to $m = 6$ dimensions, to the values of the payments we have already computed is that for $m = 5, 6$ we have that

$$p_4^{(5)} \leftarrow p_5^{(5)} \quad \text{and} \quad p_5^{(6)} \leftarrow p_6^{(6)}$$

which gives by (4.6) that also the $\mu_r^{(m)}$ parameters are updated to $\mu_{m-1}^{(m)} \leftarrow \mu_m^{(m)} - (m+1)$:

$$\mu_4^{(5)} \approx 12.0865 \qquad \mu_5^{(6)} \approx 18.3585.$$

We now introduce some parameters that will be used extensively in the following. They are the critical parameters of the SIM-bodies used in all the key theorems for the optimality of SJA, namely Lemma 4.16, Theorem 4.2 and Theorem 4.3:

$$\lambda_r \equiv \mu_r - \mu_{r-1} \tag{4.22}$$

which is equivalent to saying that $\mu_r = \lambda_1 + \cdots + \lambda_r$. Taking the $\mu_r^{(m)}$ values into account (see (4.6)) the $\lambda_r^{(m)}$'s for up to $m = 6$ items are, for $m \leq 4$:

$$\lambda_1 = 1 \qquad \lambda_2 = 1 + \sqrt{2} \qquad \lambda_3 \approx 3.6830 \qquad \lambda_4 \approx 4.9000$$

and for $m = 5, 6$ the only modifications are

$$\lambda_4^{(5)} \approx 4.9894 \qquad \lambda_5^{(5)} = 6 \qquad \lambda_5^{(6)} \approx 6.3613 \qquad \lambda_6^{(6)} = 7.$$

The non-increasing differences property of the SJA payments makes these parameters be monotonic:

Lemma 4.15. *The $\lambda_r^{(m)}$ parameters are non-decreasing and upper-bounded by $m + 1$, i.e.*

$$\lambda_{r-1}^{(m)} \leq \lambda_r^{(m)} \leq m + 1,$$

for all $r = 2, \ldots, m$.

Proof. Using the transformations (4.6) and (4.22) we have

$$p_r - p_{r-1} \leq p_{r-1} - p_{r-2} \Longrightarrow \mu_{r-1} - \mu_{r-2} \leq \mu_r - \mu_{r-1} \Longrightarrow \lambda_{r-1} \leq \lambda_r$$

and

$$p_{r-1} \le p_r \Longrightarrow \mu_r - \mu_{r-1} \le m+1 \Longrightarrow \lambda_r \le m+1,$$

which concludes the proof since the SJA payments are non-decreasing with non-increasing differences (Lemma 4.14). ☐

Now we are ready to prove Lemma 4.16, which makes rigorous the correspondence between the various components $U_J^{(m)}$ of the allocation space of SJA and SIM-bodies. It is the motivation behind introducing SIM-bodies in the first place. Essentially, the entire allocation space of SJA is made up by slices of SIM-bodies:

Lemma 4.16. *Every non-empty slice* $U_J^{(m)}\Big|_{-J:\mathbf{t}}$ *is isomorphic to the SIM-body* $k \cdot \Lambda(\lambda_1^{(m)}, \dots, \lambda_{|J|}^{(m)})$*, where* $k = \frac{1}{m+1}$.

Proof. Let $|J| = r$. Then, due to symmetry, the slice $U_J|_{-J:\mathbf{t}}$ is isomorphic to $U_{[r]}\Big|_{[r+1\dots m]:\mathbf{t}}$. An r-dimensional vector $\mathbf{y}$ belongs to this slice if and only if $(\mathbf{y}, \mathbf{t}) \in U_{[r]}$, which by Lemma 4.11 means that $\mathbf{y} \in I^r$ and

$$\bigwedge_{L \subseteq [r]} \sum_{j \in L} y_j \ge p_r - p_{r-|L|}.$$

By (4.6) this can be written as

$$\bigwedge_{L \subseteq [r]} \sum_{j \in L} y_j \ge |L| - (\mu_r - \mu_{r-|L|})k.$$

So this slice is an upwards closed body within the r dimensional unit-hypercube I^r and if we apply the isomorphism $\mathbf{y} \mapsto \mathbf{1}_r - \mathbf{y}$ it is flipped around and mapped to the downwards closed body around the origin $\mathbf{0}_r$ defined by $\mathbf{y} \in I^r$ and $\bigwedge_{L \subseteq [r]} \sum_{j \in L} y_j \le (\mu_r - \mu_{r-|L|})k$. By taking into consideration (4.22) this becomes

$$\bigwedge_{L \subseteq [r]} \sum_{j \in L} y_j \le \lambda_{r-|L|+1}k + \dots + \lambda_r k. \tag{4.23}$$

It is easy to see that the extra condition $\mathbf{y} \in I^r$ can be replaced by the weaker one $\mathbf{y} \in \mathbb{R}_+^r$, since the upper bounds $y_j \le 1$ are already captured by (4.23): for $L = \{j\}$ it gives

$$y_j \le \lambda_r k = \frac{\lambda_r}{m+1} \le 1, \tag{4.24}$$

the last inequality holding from Lemma 4.14. So, we end up with exactly the definition of $\Lambda(k\lambda_1, \dots, k\lambda_r)$. We must note here that this SIM-body is well defined, since the λ_r's are non-decreasing (Lemma 4.14). ☐

4.5 The Optimality of SJA

In this section we conclude the proof of our main result about the optimality of SJA (Theorem 4.1). Remember that parameters $\lambda_r^{(m)}$ depend on the payments of the SJA mechanism (particularly, on the $\mu_r^{(m)}$'s in (4.6)) and are given by (4.22), and that $U_J^{(m)}$ denotes the subdomain in which SJA allocates exactly the bundle $J \subseteq [m]$ of items (see (4.13)):

In addition to the SIM-bodies $\Lambda(\lambda_1, \ldots, \lambda_r)$ being essentially the building blocks of the allocation space of the SJA (Lemma 4.16), the particular choice of the λ_r parameters makes them satisfy another property; they have zero 1-deficiency:

Lemma 4.17. *For any dimension m, if a subdomain $U_J^{(m)}$ of SJA is nonempty then the corresponding SIM-body $\Lambda(\lambda_1^{(m)}, \ldots, \lambda_{|J|}^{(m)})$ has zero 1-deficiency.*

Proof. Fix some m and let $k = \frac{1}{m+1}$. For any nonempty subdomain U_J, the slice $U_J|_{-J:\mathbf{0}_{m-|J|}}$ is nonempty (by downwards closure), so by Lemma 4.12 it has zero k-deficiency. But from Lemma 4.16 it is also isomorphic to the SIM-body $k \cdot \Lambda(\lambda_1^{(m)}, \ldots, \lambda_r^{(m)})$, thus $\delta_k(k \cdot \Lambda(\lambda_1^{(m)}, \ldots, \lambda_r^{(m)})) = 0$. By Property 5 of Lemma 4.10 this means that indeed $\delta_1(\Lambda(\lambda_1^{(m)}, \ldots, \lambda_r^{(m)})) = 0$.

□

Now we are ready to prove Theorem 4.2. *It is essentially the only ingredient of this chapter whose proof does not work for more than* 6 *items* (condition (4.26), specifically). In a way it demonstrates the maximality of the deficiency of the particular critical SIM-bodies $\Lambda(\lambda_1, \ldots, \lambda_r)$, in the sense that they cannot contain subsets that have greater deficiency than themselves.

Theorem 4.2. *For up to $m \leq 6$, no SIM-body $\Lambda(\lambda_1^{(m)}, \ldots, \lambda_r^{(m)})$ corresponding to a nonempty subdomain $U_{[r]}^{(m)}$ contains positive 1-deficiency sub-bodies.*

Proof. We will prove the stronger statement that for all $r \leq m \leq 6$ no SIM-body $\Lambda(\lambda_1^{(m)}, \ldots, \lambda_r^{(m)})$ contains a sub-body with nonnegative 1-deficiency greater than its own, i.e.

$$\emptyset \neq A \subseteq \Lambda(\lambda_1^{(m)}, \ldots, \lambda_r^{(m)}) \;\; \wedge \;\; \delta_1(A) \geq 0 \quad \Longrightarrow \quad \delta_1(A) \leq \delta_1(\Lambda(\lambda_1^{(m)}, \ldots, \lambda_r^{(m)})) \tag{4.25}$$

This is enough to establish the lemma, because of Lemma 4.17. We will use induction on r. At the basis, whenever $r = 1$, for any number of items m the SIM-body is just the line segment $\Lambda(\lambda_1^{(m)}) = [0, \lambda_1^{(m)}]$ and it is easy to see that every (nonempty) subset of it will have smaller volume but the same projection, resulting to smaller deficiency.

Moving on to the inductive step, for simplicity denote $\Lambda = \Lambda(\lambda_1^{(m)}, \ldots, \lambda_r^{(m)})$ and let $A \subseteq \Lambda$ be a maximum volume sub-body of maximum nonnegative deficiency within Λ. Without loss (by Lemma 4.4) A can be assumed to be symmetric and downwards closed. By Lemma 4.9, this tells us that every slice of it must be p-closed (since A is within Λ which is a SIM-body and thus p-closed). We will prove that $A = \Lambda$ which is enough to establish (4.25).

We start by showing that the outmost $(r-1)$-dimensional slice of A, namely $A|_{\{1\}:w(\Lambda)}$, cannot be empty. Notice that, by Property 1 of Lemma 4.10, $w(\Lambda) = \lambda_r^{(m)}$. The choice of coordinate 1 here is arbitrary; due to symmetry any slice $A|_{\{j\}:w(\Lambda)}$ with $j \in [r]$ would work in exactly the same way. If this slice was empty, we could add in this free space of A (an ε-neighbourhood of) the $(r-1)$-dimensional SIM-body B defined by

$$B = \Lambda(\lambda_1^{(m')}, \ldots, \lambda_{r-1}^{(m')}) \qquad \text{where } \ m' = \begin{cases} r-1, & \text{if } U_{[r-1]}^{(m)} = \emptyset, \\ m, & \text{otherwise.} \end{cases}$$

By taking into consideration the values of the $\lambda_j^{(m)}$ parameters of the SJA mechanism we can see that the following properties are satisfied for all $j \leq m \leq 6$,

$$\lambda_j^{(m')} \leq \lambda_j^{(m)} \quad \text{and} \quad \lambda_j^{(m)} \leq j+1, \tag{4.26}$$

so it must be that

$$B \subseteq \Lambda(\lambda_1^{(m)}, \ldots, \lambda_{r-1}^{(m)}) = \Lambda|_{\{1\}:\lambda_r^{(m)}},$$

the first inclusion being a result of (4.26) and the last equality being from Property 3 of Lemma 4.10. This means that B indeed fits in the exterior space Λ at distance $x_1 = \lambda_r^{(m)}$, which is exactly where we put it.

We will now show that this addition caused no decrease at the 1-deficiency of A, which would contradict the maximality of the volume of A. Equivalently, we need to show that the increase we caused in the volume by extending A was at least equal to the increase in the total volume of its projections. First, we show that no new projections were created with respect to coordinate 1, i.e. B was already included in $A_{[r]\setminus\{1\}} = A|_{\{1\}:0}$. Indeed, it is

$$B = \mathcal{D}(\mathcal{H}(\mathcal{P}(\lambda_1^{(m')}, \ldots, \lambda_{r-1}^{(m')}))) \subseteq \mathcal{D}(\mathcal{H}(\mathcal{P}(2, \ldots, r))) \subseteq A|_{\{1\}:0}.$$

The first equality comes from Property 2 of the SIM-bodies in Lemma 4.10, the second inclusion is from (4.26) and the last inclusion is by Lemma 4.8 and the p-closure of $A|_{\{1\}:0}$. What is left to show is that the sum of the new projections created with respect to the remaining coordinates $[2...r]$ was at most equal to the increase in the volume. But this comes directly from the fact that the slice B we added has zero 1-deficiency: it is a SIM-body corresponding to a subdomain $U_{[r-1]}^{(m')} \neq \emptyset$ (see Lemma 4.17).

So, in the following we can indeed assume that body $A \subseteq \Lambda$ is of maximum width

$w(A) = \lambda_r^{(m)}$. Then we will show that at $x_1 = w(A)$, A must in fact include the *entire* corresponding slice of Λ. This slice is $\Lambda|_{\{1\}:\lambda_r^m} = \Lambda(\lambda_1^{(m)}, \ldots, \lambda_{r-1}^{(m)})$, so that would mean that the extreme point $(\lambda_1^{(m)}, \ldots, \lambda_{r-1}^{(m)}, \lambda_r^{(m)})$ is in A, and thus by p-closure (Lemma 4.9) the body $\mathcal{D}(\mathcal{H}(\mathcal{P}(\lambda_1^{(m)}, \ldots, \lambda_r^{(m)})))$ must be included within A. But from Property 2 of Lemma 4.10 this body is exactly the entire external body Λ, which concludes the proof. So let us show that indeed $A|_{\{1\}:\lambda_r^m} = \Lambda|_{\{1\}:\lambda_r^m}$. It is enough to show that removing this slice of A and replacing it with the full slice of Λ would result in a non-decrease of the 1-deficiency: that would contradict the maximality of the volume of A.

First, notice that $A|_{\{1\}:\lambda_r^m}$ is within $\Lambda|_{\{1\}:\lambda_r^m}$, where $\Lambda|_{\{1\}:\lambda_r^m}$ is the SIM-body $\Lambda(\lambda_1^{(m}, \ldots, \lambda_{r-1}^{(m)})$ and also slice $A|_{\{1\}:\lambda_r^m}$ must have nonnegative deficiency (by Lemma 4.5). So, by the induction hypothesis it must be that the full slice $\Lambda|_{\{1\}:\lambda_r^m}$ has at least the deficiency of the slice $A|_{1:\lambda_r^m}$ it replaces. That means that, taking into consideration only projections in the directions $[2...r]$, the overall change in the deficiency is indeed nonnegative. So, to conclude the proof it is enough to show that no new projections with respect to coordinate 1 are created by this replacement, i.e. that $\Lambda(\lambda_1^{(m}, \ldots, \lambda_{r-1}^{(m)})$ was already included in $A_{[r]\setminus\{1\}} = A|_{\{1\}:0}$. Indeed:

$$\begin{aligned}
\Lambda(\lambda_1^{(m}, \ldots, \lambda_{r-1}^{(m)}) &= \mathcal{D}(\mathcal{H}(\mathcal{P}(\lambda_1^{(m)}, \ldots, \lambda_{r-1}^{(m)}))) \\
&\subseteq \mathcal{D}(\mathcal{H}(\mathcal{P}(\lambda_1^{(m)}, \ldots, \lambda_{r-2}^{(m)}, \lambda_r^{(m)}))) \\
&\subseteq \mathcal{D}(\mathcal{H}(\mathcal{P}(2, \ldots, r-1, w(A)))),
\end{aligned}$$

by (4.26) and the fact that $w(A) = \lambda_r^{(m)}$, which concludes the proof since slice $A|_{\{1\}:0}$ is p-closed and $(2, \ldots, r-1, w(A))$ belongs to it, because $(1, 2, \ldots, r-1, w(A))$ belongs to A by Lemma 4.8. □

We now present our main tool to prove that SJA is optimal. It utilizes the fact that the allocation space of SJA has no positive deficiency subsets in a combinatorial way.

Theorem 4.3. *If for every nonempty subdomain $U_J^{(m)}$ of SJA the corresponding SIM-body $\Lambda(\lambda_1^{(m)}, \ldots, \lambda_{|J|}^{(m)})$ contains no sub-bodies of positive 1-deficiency, then SJA is optimal.*

Proof. The proof of Theorem 4.3 is done via a combinatorial detour to a discrete version of the problem, which is interesting in its own right and highlights the connection of the dual program with bipartite matchings. The nonpositive deficiencies property allows us to utilize Hall's marriage condition. Let us denote by $I_j \equiv \{(\mathbf{x}_{-j}, 1) \mid \mathbf{x} \in I^m\}$ the side on the boundary of the I^m cube which is perpendicular to axis j, for $j \in [m]$.

We start by restricting the search for an appropriate feasible dual solution to those functions $z_j(\mathbf{x})$ that have the following form:

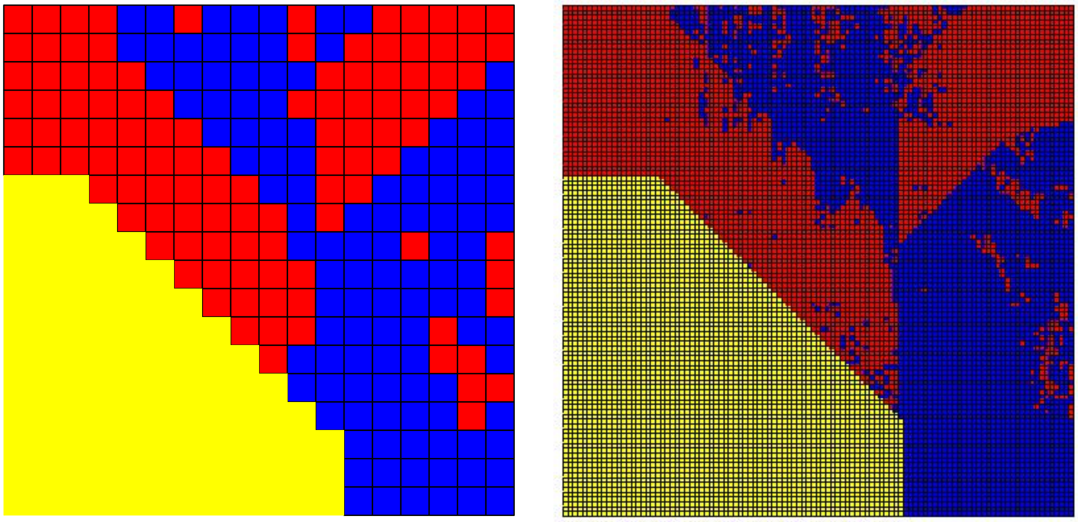


Figure 4.4: Proper colourings of the allocation space $\overline{U}_\emptyset$ of the SJA mechanism for $m = 2$ items and different discretization factors $N = 18$ (left) and $N = 105$ (right). Blue corresponds to the direction of the horizontal axis and red to the vertical axis. The zero region $U_\emptyset$ where no item is allocated (white region in Figure 4.2) is coloured in yellow. Notice how the entire region $U_{\{1\}}$ is coloured blue and the entire $U_{\{2\}}$ red. The critical and technically involved part of the colouring for two items is the one of region $U_{\{1,2\}}$ where both items are allocated. Interpreting this in the realm of the dual program and the language of the proof of Theorem 4.3, blue is colour 1 and corresponds to the points where function z_1 increases with "full" derivative $m + 1 = 3$ (with respect to the coordinate x_1) while z_2 remains constant (with respect to coordinate x_2). Red is colour 2 and denotes the reverse situation where z_2 increases with derivative 3 (with respect to coordinate x_2) and z_1 remains constant (with respect to variable x_1). Yellow is colour 0 where both z_1 and z_2 are constant.

> Fix some integer N which is a multiple of $m + 1$ and let $\varepsilon' = 1/N$. We discretize the space by taking a fine grid partition of the hypercube I^m into small hypercubes of side ε' and we require that inside each small hypercube the derivatives $\partial z_j(\mathbf{x})/\partial x_j$ are constant and take either value 0 or value $m + 1$.

We must point out here that this discretization is used only in the analysis and it is not part of the optimal selling mechanism which is given just by its prices $p_r^{(m)}$.

With the discretization, the combinatorial nature of the dual solutions emerges: a dual solution is essentially a colouring of all the ε'-hypercubes of I^m into colours $0, 1, \ldots, m$. The interpretation of the colouring is the following: the derivative $\partial z_j(\mathbf{x})/\partial x_j$ has a positive value $m + 1$ if and only if the corresponding hypercube (at which $\mathbf{x}$ belongs to) has colour j, otherwise it is zero (i.e. $z_j(\mathbf{x})$ is constant with respect to the direction of the j-axis); colour 0 is used exactly for the points where *all* z_j functions are constant. A *feasible* dual solution corresponds to a colouring in which every line of hypercubes parallel to some axis, say axis j, contains at least $N/(m+1)$ hypercubes of colour j. To see this, notice that function $z_j(\mathbf{x})$ must increase in a fraction of (at least) $1/(m + 1)$ of those small hypercubes (because it starts at value 0 and has to increase to a value of at least 1, see the dual constraints in Theorem 6.1). Figure 4.4 illustrates such a colouring for $m = 2$ items.

To formalize this let us discretize the unit-cube I^m in ε'-hypercubes

$$[(i_1 - 1)\cdot\varepsilon', i_1\cdot\varepsilon'] \times \cdots \times [(i_m - 1)\cdot\varepsilon', i_m\cdot\varepsilon']$$

where $i_j \in [N]$ for all $j \in [m]$ (see Figure 4.5). To keep notation simple, we will sometimes identify hypercubes by their centre points, i.e. refer to the ε'-hypercube $\mathbf{x}$ instead of the cube $[x_1 - \varepsilon'/2, x_1 + \varepsilon'/2] \times \cdots \times [x_m - \varepsilon'/2, x_m + \varepsilon'/2]$. In that way, I^m is essentially an m-dimensional lattice of points

$$((i_1 - 1)\cdot\varepsilon' + \varepsilon'/2, \ldots, (i_m - 1)\cdot\varepsilon' + \varepsilon'/2)\,, \quad i_j \in [N], j \in [m].$$

Based on this, for any $S \subseteq I^m$ we will denote by $\Delta(S)$ the set of lattice points in S.

Next, consider the subdomain U_J where SJA sells exactly the items that are in $J \subseteq [m]$. For any one of these "active" coordinates $j \in J$ take U_J's boundary at side I_j of the unit-cube and "inflate" it to have a width of $k = \frac{1}{m+1}$. Formally, for all $J \subseteq [m]$ and $j \in J$ define

$$B_{J,j} \equiv \{(t, \mathbf{x}_{-j}) \mid \mathbf{x} \in U_J \;\wedge\; t \in [1, 1+k]\}\,. \tag{4.27}$$

$B_{J,j}$ is isomorphic to $(U_J)_{[m]\setminus\{j\}} \times [0, k]$. For any subset of items $J \subseteq [m]$ denote $B_J = \bigcup_{j\in J} B_{J,j}$ and $B = \bigcup_{J\subseteq[m]} B_J$ the entire external layer on all sides.

Notice that $\overline{U}_\emptyset$ cannot be perfectly discretized: the small hypercubes do not fit exactly inside $\overline{U}_\emptyset$ because its boundaries are not rectilinear[4]. To fix this, we will take a cover $\overline{U}^*_\emptyset$ of $\overline{U}_\emptyset$ which *can* be partitioned into ε'-hypercubes. More precisely, define $\overline{U}^*_\emptyset$ to be the union of all ε'-hypercubes of I^m that intersect $\overline{U}_\emptyset$. Finally, let us also extend the boundary region B by adding on top of every boundary component $B_{J,j}$ a thin strip

$$B^*_{J,j} = \{(t, \mathbf{x}_{-j}) \mid \mathbf{x} \in U_J \;\wedge\; t \in [1+k, 1+k+g(m)\cdot\varepsilon']\} \tag{4.28}$$

where $g(m) = \lceil\sqrt{m}+1\rceil$, and extend notation in the obvious way: $B^*_J = \bigcup_{j\in J} B^*_{J,j}$ and $B^* = \bigcup_j B^*_j$.

Now it's time to fully reveal the combinatorial structure of our construction by defining a bipartite graph $G(\Delta(\overline{U}^*_\emptyset) \cup \Delta(B \cup B^*), E)$, which has as nodes the ε'-hypercubes of the cover $\overline{U}^*_\emptyset$ and the boundary $B \cup B^*$ (see Figure 4.5). Intuitively, the edges E will connect all lattice points of a subdomain U_J with the nodes of its corresponding boundary $B_J \cup B^*_J$ that agree on $m-1$ coordinates; each U_J is projected onto the sides I_j of the cube that correspond to active items $j \in J$. To be precise, for any $\mathbf{x} \in \Delta(\overline{U}^*_\emptyset)$ and $\mathbf{y} \in \Delta(B \cup B^*)$,

$$(\mathbf{x}, \mathbf{y}) \in E \iff \mathbf{x}_{-j} = \mathbf{y}_{-j} \text{ for some } j \in J,\ J \subseteq [m], \text{ with } \varepsilon'\text{-hypercube } \mathbf{x} \text{ intersecting } U_J.$$

[4]The solution of partitioning the unit hypercube into small simplices instead of small hypercubes does not work either; although simplices have more appropriate boundaries, we cannot guarantee that exists an ε' for which all the boundaries of $\overline{U}_\emptyset$ coincide with some boundaries of the small simplices.

Another way to view this is that edges start from a node on a side j of the external layer $B \cup B^*$, are perpendicular to that side of the unit-cube (i.e. parallel to axis j) and run towards its interior body $\overline{U}_\emptyset^*$, excluding the areas where j is not sold.

By this construction, *a bipartite matching of graph G that matches* completely *the initial boundary $\Delta(B)$ corresponds to a proper colouring of the ε'-hypercubes of I^m*: an internal cube matched to a node in side B_j is assigned colour j and all unmatched cubes are assigned colour 0; every line parallel to an axis $j \in [m]$ contains at least $k/\varepsilon' = N/(m+1)$ distinct hypercubes in the boundary B_j.

What does the nonpositive 1-deficiency property of all SIM-bodies $\Lambda(\lambda_1, \ldots, \lambda_r)$, $r \leq m$, can tell us about graph G? Remember (Lemma 4.16) that these SIM-bodies correspond to slices $U_J|_{-J:\mathbf{t}}$ of the allocation space, so (using also Property 5 of Lemma 4.10) for any $J \subseteq [m]$, $\mathbf{t} \in \mathbb{R}_+^{m-|J|}$ and $S \subseteq U_J|_{-J:\mathbf{t}}$:

$$|S| \leq k \sum_{j \in J} |S_j| \,, \tag{4.29}$$

where $S_j \equiv S_{[m]\setminus\{j\}}$. Using the fact that every such slice $U_J|_{-J:\mathbf{t}}$ has zero k-deficiency (Lemma 4.12), if we take compliments

$$\overline{S} = U_J|_{-J:\mathbf{t}} \setminus S \quad \text{and} \quad \overline{S}_j = \left(U_J|_{-J:\mathbf{t}}\right)_{[m]\setminus\{j\}} \setminus S_j$$

the above relation gives

$$k \sum_{j \in J} \left|\overline{S}_j\right| \leq \left|\overline{S}\right| . \tag{4.30}$$

First we will show that there is a matching on the bipartite graph G we defined, which completely matches all nodes in $\Delta(\overline{U}_\emptyset)$. By (4.29) and the fact that every $(U_J)_{[m]\setminus\{j\}} \times [0, k]$ is isomorphic to $B_{J,j}$, Hall's theorem tells us that we can completely match $\Delta(U_J|_{-J:\mathbf{t}})$ into $\Delta(B_J)$. By the way we have constructed the edge set E, this directly means that there is a complete matching of $\Delta(\overline{U}_\emptyset)$ into $\Delta(B)$. So, to extend this into a complete matching of the cover $\Delta(\overline{U}_\emptyset^*)$, it is enough to show that the extra lattice points in $\overline{U}_\emptyset^* \setminus \overline{U}_\emptyset$ of any line parallel to some axis j are at most $g(m)$, the number of neighbours in the extended thin-stripe boundary B^*. Indeed, any point in $\overline{U}_\emptyset^*$ cannot have distance more than $\sqrt{m}\varepsilon' \leq g(m)\varepsilon'$ from a point in $\overline{U}_\emptyset$, because every ε'-hypercube of $\overline{U}_\emptyset^*$ intersects with $\overline{U}_\emptyset$ and the diameter of such a hypercube (with respect to the euclidean metric) is exactly $\sqrt{m}\varepsilon'$.

We will now show that there is also a complete matching of $\Delta(B)$ into $\Delta(\overline{U}_\emptyset^*)$. By the way we constructed the edge set, it is enough to show that every slice $\Delta(B_J|_{-J:\mathbf{t}})$ of the boundary can be completely matched into the corresponding internal slice $\Delta(\overline{U}_\emptyset^*|_{-J:\mathbf{t}})$. Fix some nonempty $J \subseteq [m]$ and $\mathbf{t} \in \mathbb{R}_+^{m-|J|}$. By Hall's Theorem (see Section 4.3.3) it is

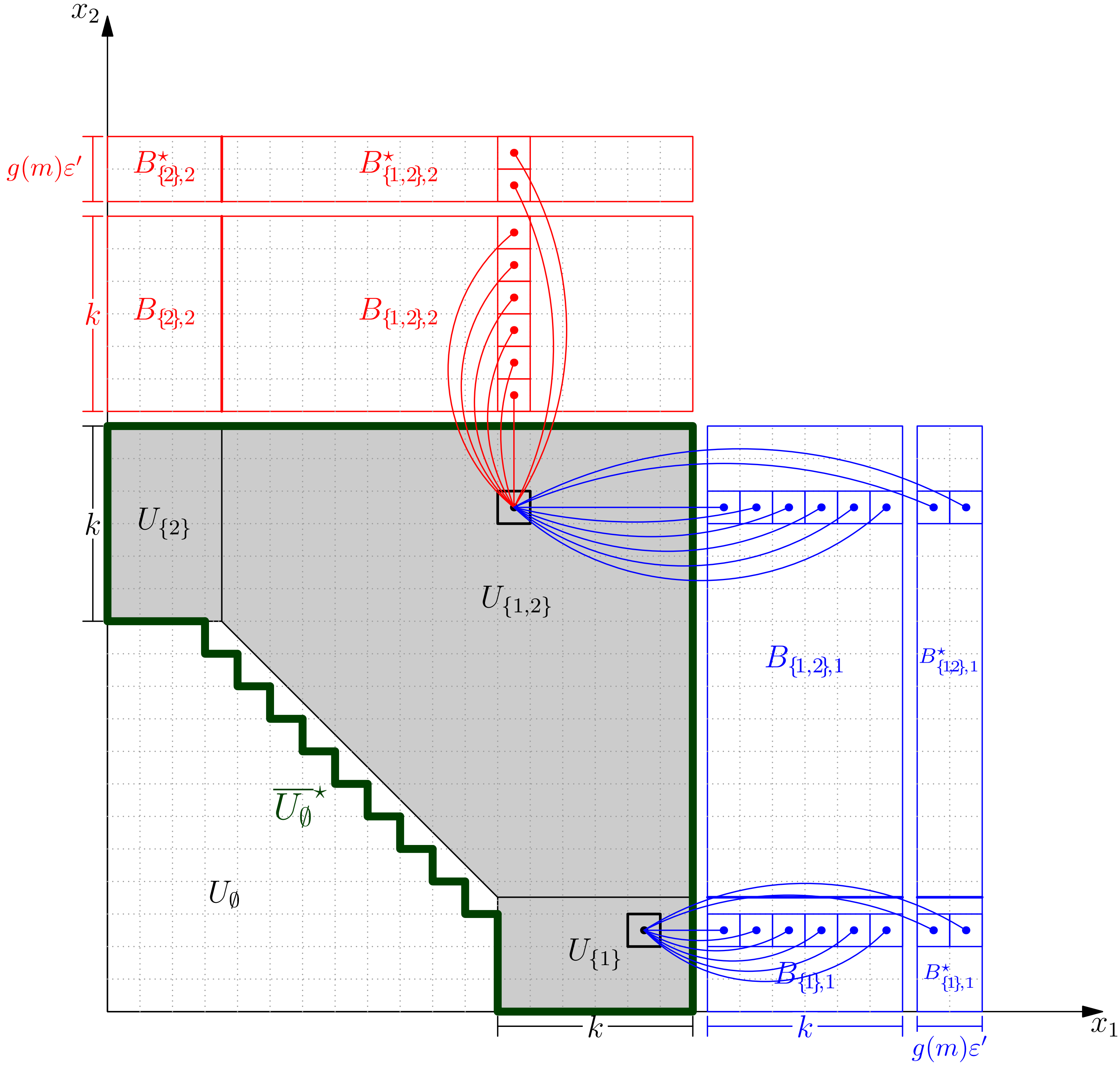


Figure 4.5: The discretization of the allocation space and the structure of graph G used in the proof of Theorem 4.3, for $m = 2$ items. The space $\overline{U}_\emptyset$ where SJA sells at least one item (coloured grey) does not properly align with the ε'-discretization grid so we have to take a cover $\overline{U}_\emptyset^*$ (outlined with the thick line, green in the colour version of this thesis). The boundaries $B_j \cup B_j^*$ have width $k+g(m)\varepsilon'$. The one on the right (perpendicular to the vertical axis) consists of ε'-cubes holding colour 1 (blue at the colour version of the thesis) and the one at the top colour 2 (red). Edges run from every internal ε'-cube, vertically towards the red exterior and horizontally towards the blue exterior. Notice, however, how the cube within the allocation subspace $U_{\{1\}}$ has only horizontal (blue) edges running out of it, since it is not allowed to use colour 2 (red). That is due to the fact that item 2 is not sold within $U_{\{1\}}$.

enough to prove that for any family of sets of $\{T_j\}_{j\in J}$ of lattice points $T_j \subseteq \Delta(B_{J,j}|_{-J:\mathbf{t}})$:

$$\sum_{j\in J} |T_j| \leq |\bigcup_{j\in J} N(T_j)|.$$

We will prove the stronger

$$\sum_{j\in J} |T_j| \leq |\bigcup_{j\in J} N(T_j) \cap \overline{U}_\emptyset|_{-J:\mathbf{t}}|,$$

that is, we will just count neighbours in the initial set $\overline{U}_\emptyset$ and not the cover $\overline{U}^*_\emptyset$. The continuous analogue of this is to take T_j's be subsets of $B_{J,j}|_{-J:\mathbf{t}}$ and consider the natural extension of the neighbour function N when we now have a infinite graph of edges

$$\left\{(\mathbf{x},\mathbf{y}) \;\middle|\; \mathbf{x} \in U_J|_{-J:\mathbf{t}} \;\wedge\; \mathbf{y} \in B_J|_{-J:\mathbf{t}} \;\wedge\; \mathbf{x}_{-j} = \mathbf{y}_{-j} \text{ for some } j \in J\right\}$$

Let

$$S = U_J|_{-J:\mathbf{t}} \setminus \bigcup_{j\in J} N(T_j)$$

be the set of points *not* being neighbours of any node in $\cup_{j\in J} T_j$ of the boundary. Then by (4.30) it is enough to show that

$$\sum_{j\in J} |T_j| \leq k \sum_{j\in J} \left|\overline{S}_j\right|,$$

where $S_j = S_{[m]\setminus\{j\}}$. Every point in the boundary $B_{J,j}|_{-J:\mathbf{t}}$ that has neighbours in $\bigcup_{j\in J} N(T_j)$ projects (with respect to j) inside $\overline{S}_j$. But, for any point $\mathbf{y}$ in T_j the only other points that can have the same projection with respect to coordinate j are all points of the line segment of $B_{J,j}|_{-J:\mathbf{t}}$ which is parallel to the j-axis and passes through $\mathbf{y}$, and this segment has length k.

Combining the existence of the above two matchings, a straightforward use of the classic Cantor-Bernstein theorem from Set Theory ensures the existence of a matching in graph G that *completely* matches both $\Delta(\overline{U}^*_\emptyset)$ and $\Delta(B)$. But as we discussed before, this means that $\overline{U}^*_\emptyset$ is properly colourable and thus, this colouring induces a feasible dual solution. Let's denote this solution by $z_j(\mathbf{x})$, $j \in [m]$ and also let $u(\mathbf{x})$ be the primal solution given by SJA, i.e. u is the utility function of the SJA mechanism. To prove the optimality of u, we will take advantage of the *approximate* complementarity: we claim that this primal-dual pair of solutions satisfies the approximate complementarity conditions in Lemma 3.2 for $\varepsilon = g(m)m(m+1)\cdot\varepsilon'$:

$$u(\mathbf{x})\cdot\left(m+1-\sum_{j\in[m]}\frac{\partial z_j(\mathbf{x})}{\partial x_j}\right)\le\varepsilon \tag{4.31}$$

$$-u(0,\mathbf{x}_{-j})\cdot z_j(0,\mathbf{x}_{-j})\le\varepsilon \tag{4.32}$$

$$u(1,\mathbf{x}_{-j})\cdot(z_j(1,\mathbf{x}_{-j})-1)\le\varepsilon \tag{4.33}$$

$$z_j(\mathbf{x})\cdot\left(1-\frac{\partial u(\mathbf{x})}{\partial x_j}\right)\le\varepsilon, \tag{4.34}$$

If that is true, then the proof of Theorem 4.3 is complete, since by the approximate complementarity Lemma 3.2 the primal and dual objectives differ by at most $(3m+1)\varepsilon = (3m+1)g(m)m(m+1)\varepsilon'$ and if we take the limit of this as $\varepsilon'\to 0$, these values must be equal. So let us prove that (4.31)–(4.34) indeed hold.

Condition (4.32) is satisfied trivially, since both the primal and the dual variables are nonnegative. Regarding (4.33), for any line parallel to some axis j the length of its segment intersecting the boundary $B\cup B^*$ (which is the one contributing the critical colours j to that direction) is $k+g(m)\varepsilon$. So, given that the derivative of $z_j(\mathbf{x})$ in sections coloured with j is $m+1$ we can upper-bound the value of $z_j(1,\mathbf{x}_j)$ by $(k+g(m)\varepsilon')(m+1) = 1+g(m)(m+1)\varepsilon'$. This means that $z_j(1,\mathbf{x}_j)-1\le g(m)(m+1)\varepsilon'$ and given the fact that the utility function has the property that $u(\mathbf{x})\le m$ (because its derivatives are at most 1 at every direction), we finally get the desired

$$u(1,\mathbf{x}_{-j})\cdot(z_j(1,\mathbf{x}_{-j})-1)\le g(m)m(m+1)\varepsilon'=\varepsilon.$$

For condition (4.31), assume that $u(\mathbf{x})>0$ (otherwise it is satisfied). That means that SJA sells at least one item, thus $\mathbf{x}\in\overline{U}_\emptyset\subseteq\overline{U}^*_\emptyset$; but $\overline{U}^*_\emptyset$ is completely matched, thus all points of $\overline{U}^*_\emptyset$ are coloured with some colour in $[m]$ (not with colour 0); this is equivalent to the fact that some derivative of the z_j functions is $m+1$ and all others are zero, meaning that the corresponding slack variable $m+1-\sum_{j\in[m]}\partial z_j(\mathbf{x})/\partial x_j$ is zero.

Finally, for condition (4.34), fix some direction $j\in[m]$ and assume that $\partial u(\mathbf{x})/\partial x_j\neq 1$ (otherwise the condition is satisfied). SJA is deterministic, so it must be that $\partial u(\mathbf{x})/\partial x_j = 0$, i.e. item j is not allocated. That means that $\mathbf{x}$ belongs to a sub-domain U_J with $j\notin J$, and the same is true for all points before it parallel to axis j (that is, all points $(t,\mathbf{x}_{-j})$ with $t\in[0,x_j]$). Thus, by the way that the edge set E of the graph G was defined, $\mathbf{x}$'s ε'-hypercube, as well as all hypercubes before it and parallel to axis j, cannot have been coloured with colour j unless they happen to intersect with a neighbouring sub-domain U_{J^*} with $j\in J^*$. But it is a simple geometric argument to see that point $(x_j-\varepsilon' m,\mathbf{x}_{-j})$ is at distance at least $\frac{\varepsilon' m}{\sqrt{|J^*|}}\ge\frac{\varepsilon' m}{\sqrt{m}}=\sqrt{m}\varepsilon'$ below the boundary $\sum_{j\in J^*}x_j = p_{|J^*|}$ of U_{J^*} (since we already know that $\mathbf{x}$ is below it), which is exactly the diameter of the ε-hypercubes. So, at most m such hypercubes below

$\mathbf{x}$'s could intersect with U_{J^*}, and thus be coloured with colour j, meaning that $z_j(\mathbf{x})$ cannot have increased more than $(m+1)\varepsilon' \cdot (m+1)$ from zero. This proves that indeed

$$z_j(\mathbf{x})(1 - \partial u(\mathbf{x})/\partial x_j) = z_j(\mathbf{x}) \leq (m+1)^2\varepsilon' \leq \varepsilon.$$

□

Chapter 5

The Case of Two Items

In this chapter we focus on the case of a seller with only two goods facing an (additive) bidder whose values for the items come from independent (but not necessarily identical) distributions. The restriction in the number of items will allow us for more flexibility with respect to the distributional priors, going this time way beyond the uniform valuations of Chapter 4. We consider distributions supported over closed intervals of the form $[0, b]$, since we believe this to demonstrate greater flexibility: most results for infinite supports $[0, \infty)$ can be recovered simply by taking $b \to \infty$, while the inverse is not true. For example, as we will see in Corollary 5.2, for two goods with values following the (truncated) exponential distribution over $[0, 1]$, the optimal mechanism is a *randomized* one, contradicting the case of the unbounded support where from the work of Daskalakis et al. [25] we know that the simple, deterministic full-bundling selling mechanism is optimal. Here, as in the previous Chapter 4, we are primarily interested in *exact* optimality results, although a discussion of an interesting approximation technique which we refer to as *convexification* is being also made in Section 5.5.

Although the conditions that the probability distributions must satisfy are quite general, they leave out a large class of distributions. For example, they do not apply to power-law distributions with parameter $\alpha > 2$. In other words, this work goes some way towards the complete solution for arbitrary distributions for two items, but the general problem is still open. We opted towards simple conditions rather than full generality, but we believe that extensions of our method can generalize significantly the range of distributions; we expect that a proper "ironing" procedure will enable our technique to resolve the general problem for two items.

We introduce general but simple and clear, closed-form distributional conditions that can guarantee optimality and immediately give the form of the revenue-maximizing selling mechanism (its payment and allocation rules), for the setting of two goods with valuations distributed over bounded intervals (Theorem 5.1). For simplicity and a clearer exposition we study distributions supported over the real unit interval $[0, 1]$. By scaling, the results generalize immediately to intervals that start at 0, but more work would be needed to generalize them to arbitrary intervals. We use the closed forms to get optimal solutions for a wide class of distributions satisfying certain simple analytic assumptions (Theorem 5.2 and Section 5.4.2). As useful examples we provide exact solutions for families of monomial ($\propto x^c$) and exponential ($\propto e^{-\lambda x}$) distributions (Corollaries 5.1 and 5.2 and Section 5.4.2), and also near-optimal results for power-law ($\propto (x+1)^{-\alpha}$) distributions (Section 5.5). This last approximation is an application of a more general result (Theorem 5.3) involving the relaxation of some of the conditions for optimality in the main Theorem 5.1; the "solution" one gets in this new setting might not always correspond to a feasible selling mechanism, however it still provides an upper bound on the optimal revenue as well as hints as to how to design a well-performing mechanism, by "convexifying" it into a feasible mechanism (Section 5.5). Particularly for the family of monomial distributions it turns out that the optimal mechanism is a very simple deterministic mechanism that offers to the seller a menu of size complexity [39] just 4: fixed prices for each one of the two items and for their bundle, as well as the option of not buying any of them. For the rest of the distributions randomization is essential for optimality, as is generally expected in such problems of multidimensional revenue maximization (see e.g. [41, 66, 25]).

Techniques The main result of this chapter (Theorem 5.1) is proven by utilizing the duality framework we developed in Chapter 3 for revenue maximization, and in particular using complementarity: the optimality of the proposed selling mechanism is shown by verifying the existence of a dual solution with which they satisfy together the required complementary slackness conditions of the duality formulation. For clarity we state the main duality tools adapted to our two-item case in Section 5.2. Constructing these dual solutions explicitly seems to be a very challenging task and in fact there might not even be a concise way to do it, especially in closed-form. So instead we just prove the *existence* of such a dual solution, like we did in the previous Chapter 4, but here using a *max-flow min-cut* argument instead as main tool (Lemma 5.3, Figure 5.2). This is, in a way, an abstraction of the technique followed in Chapter 4, which was based on Hall's theorem for bipartite matchings. Since here we are dealing with general and non-identical distributions, this kind of refinement is essential and non-trivial, and in fact forms the most technical part of the chapter. Our approach has a strong geometric flavour, enabled again by the notion of the deficiency (see previous Section 4.3.1) of a two-dimensional body (Lemma 5.2), which is inspired by classic matching theory [64, 51].

5.1 The Model

As we mentioned in the introduction, in this chapter we assume two independently distributed goods with types x_1, x_2 following distributions F_1, F_2, respectively, with absolutely continuous densities f_1, f_2 over the unit interval I, respectively. We now present the conditions on the probability distributions which enable our technique to provide a closed-form of the optimal auction.

Assumption 5.0. The density functions f_1, f_2 are bounded from below, except for small values of x_i; in particular, we assume that there exists some small ϵ such that $f_i(x_i) > \epsilon$, for every $x_i > \epsilon$.

Assumption 5.1. The probability distributions F_1, F_2 are such that functions $h_{f_1,f_2}(\mathbf{x}) - f_2(1)f_1(x_1)$ and $h_{f_1,f_2}(\mathbf{x}) - f_1(1)f_2(x_2)$ are nonnegative, where

$$h_{f_1,f_2}(\mathbf{x}) \equiv 3f_1(x_1)f_2(x_2) + x_1 f_1'(x_1)f_2(x_2) + x_2 f_2'(x_2)f_1(x_1). \tag{5.1}$$

Function h_{f_1,f_2} will also be assumed to be absolutely continuous with respect to each of its coordinates.

We will drop the subscript f_1, f_2 in the above notations whenever it is clear which distributions we are referring to. Assumption 5.1 is a slightly stronger condition than the common regularity assumption $h(\mathbf{x}) \geq 0$ (see Equation (2.17)) in the economics literature for multidimensional auctions. In fact, Manelli and Vincent [53] make the even stronger assumption that for each item j, $x_j f_j(x_j)$ is an increasing function. Even more recently, that assumption has also been deployed by Wang and Tang [79] in a two-item setting as one of their sufficient conditions for the existence of optimal auctions with small-sized menus.

Strengthening the regularity condition $h(\mathbf{x}) \geq 0$ to that of Assumption 5.1 is essentially only used as a technical tool within the proof of Lemma 5.2, and as a matter of fact we don't really need it to hold in the entire unit box I^2 but just in a critical subregion $D_{1,2}$ which corresponds to the valuation subspace where both items are sold with probability 1 (see Figure 5.1 and Section 5.3.2). The same is true for Assumption 5.0, which is used in the proof of Lemma 5.3. As mentioned earlier in the Introduction, we introduce these technical conditions in order to simplify our exposition and enforce the clarity of the techniques, but we believe that a proper "ironing" [58] process can probably bypass these restrictions and generalize our results.

The critical Assumption 5.1 is of course satisfied by all distributions considered in the results of this paper, namely monomial $\propto x^c$ for any power $c \geq 0$ (Corollary 5.1), exponential $\propto e^{-\lambda x}$ with rates $\lambda \leq 1$ (Corollary 5.2), power-law $\propto (t+1)^{-\alpha}$ with parameters $\alpha \leq 2$ (Example 2), as well as combinations of these (see Example 1). However, there is still a large class of distributions not captured by Assumption 5.1 as it is, e.g. exponential with rates larger than 1, power-law with parameters greater than 2 and some beta-distributions (take, for example, $\propto x^2(1-x)^2$). See Footnote 2 for an alternative condition that can replace Assumption 5.1.

5.2 The Primal and Dual Programs

The major underlying tool to prove the main result in this chapter, Theorem 5.1, will be our duality framework of Chapter 3. For ease of reference and completeness, we briefly present here the formulation specialized for our case of two items. As we did in Chapter 4, and discussed in Section 3.2.2, we will again further relax the primal Program (3.1) by dropping the positive derivatives constraints.

Remember that the revenue optimization problem we want to solve is

$$\text{maximize} \qquad \mathcal{R}(u; F_1 \times F_2) \equiv \int_0^1 \int_0^1 \left(\frac{\partial u(\mathbf{x})}{\partial x_1} + \frac{\partial u(\mathbf{x})}{\partial x_1} - u(\mathbf{x}) \right) f_1(x_1) f_2(x_2) \, d\mathbf{x}$$

over the space of absolutely continuous functions $u : I^2 \longrightarrow \mathbb{R}_+$ having the properties

$$\frac{\partial u(\mathbf{x})}{\partial x_1}, \frac{\partial u(\mathbf{x})}{\partial x_2} \leq 1, \tag{5.2}$$

for a.e. $x_1, x_2 \in I$, and the dual becomes

$$\text{minimize} \int_0^1 \int_0^1 z_1(\mathbf{x}) + z_2(\mathbf{x}) \, d\mathbf{x}$$

over the space of absolutely continuous functions $z_1, z_2 : I^2 \longrightarrow \mathbb{R}_+$ with

$$z_j(0, x_{-j}) = 0, \qquad j = 1, 2, \tag{5.3}$$

$$z_j(1, x_{-j}) \geq f_j(1) f_{-j}(x_{-j}), \qquad j = 1, 2, \tag{5.4}$$

$$\frac{\partial z_1(\mathbf{x})}{\partial x_2} + \frac{\partial z_2(\mathbf{x})}{\partial x_2} \leq h(\mathbf{x}). \tag{5.5}$$

for a.e. $x_1, x_2 \in I$, where h is defined in (5.1).

Intuitively, every dual solution z_j must start at zero and grow all the way up to $f_j(1) f_{-j}(x_{-j})$ while travelling in interval I, in a way that the sum of the rate of growth of both z_1 and z_2 is never faster than the right hand side of (5.5). In Chapter 3 we showed that indeed these two programs satisfy both weak duality, i.e. for any feasible u, z_1, z_2 we have

$$\mathcal{R}(u; F_1 \times F_2) \leq \int_0^1 \int_0^1 z_1(\mathbf{x}) + z_2(\mathbf{x}) \, d\mathbf{x}$$

as well as complementary slackness, in the form of the even stronger following form of ε-complementarity:

Lemma 5.1 (Complementarity for two items)**.** *If u, z_1, z_2 are feasible primal and dual solutions, respectively, $\varepsilon > 0$ and the following complementarity constraints hold for* *a.e.* $\mathbf{x} \in I^2$,

$$u(\mathbf{x})\left(h(\mathbf{x}) - \frac{\partial z_1(\mathbf{x})}{\partial x_1} - \frac{\partial z_2(\mathbf{x})}{\partial x_2}\right) \leq \varepsilon f_1(x_1) f_2(x_2), \tag{5.6}$$

$$u(1, x_{-j})\left(z_j(1, x_{-j}) - f_j(1) f_{-j}(x_{-j})\right) \leq \varepsilon f_j(1) f_{-j}(x_{-j}), \qquad j = 1, 2, \tag{5.7}$$

$$z_j(\mathbf{x})\left(1 - \frac{\partial u(\mathbf{x})}{\partial x_j}\right) \leq \varepsilon f_1(x_1) f_2(x_2), \qquad j = 1, 2, \tag{5.8}$$

where h is defined in (5.1), then the values of the primal and dual programs differ by at most 7ε. In particular, if the conditions are satisfied with $\varepsilon = 0$, both solutions are optimal.

5.3 Sufficient Conditions for Optimality

This section is dedicated to proving the main result of the chapter:

Theorem 5.1. *If there exist decreasing, concave functions $s_1, s_2 : I \to I$, with $s_1'(t), s_2'(t) > -1$ for all $t \in I$, such that for almost every $x_1, x_2 \in I$*

$$\frac{s_1(x_2) f_1(s_1(x_2))}{1 - F_1(s_1(x_2))} = 2 + \frac{x_2 f_2'(x_2)}{f_2(x_2)} \quad \text{and} \quad \frac{s_2(x_1) f_2(s_2(x_1))}{1 - F_2(s_2(x_1))} = 2 + \frac{x_1 f_1'(x_1)}{f_1(x_1)}, \tag{5.9}$$

then there exists a constant $p \in [0, 2]$ such that

$$\int_D h(\mathbf{x})\, dx_1\, dx_2 = f_1(1) + f_2(1) \tag{5.10}$$

where D is the region of I^2 enclosed by curves[1] $x_1 + x_2 = p$, $x_1 = s_1(x_2)$ and $x_2 = s_2(x_1)$ and including point $(1, 1)$, i.e.

$$D = \left\{\mathbf{x} \in I \;\middle|\; x_1 + x_2 \geq p \vee x_1 \geq s_1(x_2) \vee x_2 \geq s_2(x_1)\right\},$$

and the optimal selling mechanism is given by the utility function

$$u(\mathbf{x}) = \max\left\{0, x_1 - s_1(x_2), x_2 - s_2(x_1), x_1 + x_2 - p\right\}. \tag{5.11}$$

In particular, if $p \leq \min\{s_1(0), s_2(0)\}$, then the optimal mechanism is the deterministic full-bundling with price p.

[1] See Figure 5.1.

Notice that for any $s \in I$ we have

$$\begin{aligned}
\int_s^1 h(\mathbf{x})\,dx_1 &= \int_s^1 3f_1(x_1)f_2(x_2) + x_1 f_1'(x_1)f_2(x_2) + x_2 f_2'(x_2)f_1(x_1)\,dx_1 \\
&= 3f_2(x_2)(1-F_1(s)) + f_2(x_2)\int_s^1 x_1 f_1'(x_1)\,dx_1 + x_2 f_2'(x_2)(1-F_1(s)) \\
&= 3f_2(x_2)(1-F_1(s)) + f_2(x_2)\left([x_1 f_1(x_1)]_s^1 - (1-F_1(s))\right) + x_2 f_2'(x_2)(1-F_1(s)) \\
&= 2f_2(x_2)(1-F_1(s)) + f_2(x_2)(f_1(1) - s f_1(s)) + x_2 f_2'(x_2)(1-F_1(s)) \\
&= (1-F_1(s))f_2(x_2)\left[2 + \frac{x_2 f_2'(x_2)}{f_2(x_2)} - \frac{s f_1(s)}{1-F_1(s)}\right] + f_1(1)f_2(x_2)
\end{aligned}$$

which means that an equivalent way of looking at (5.9) is, more simply, by

$$\int_{s_1(x_2)}^1 h(\mathbf{x})\,dx_1 = f_1(1)f_2(x_2) \quad \text{and} \quad \int_{s_2(x_1)}^1 h(\mathbf{x})\,dx_2 = f_2(1)f_1(x_1). \tag{5.12}$$

5.3.1 Flows

To carry out the proof of Theorem 5.1 we will need to use some basic notions and property of flow networks, and we very briefly present them here for completeness. Assume a directed graph $G = (V, E)$, together with a capacity function $c : E \longrightarrow \mathbb{R}_+$ over the edges. Also, there are two special nodes $s, t \in V$, called the *source* and *sink* (or destination), respectively. The in-degree of s and the out-degree of t are both zero. Then, a *flow* over G is a function $f : E \longrightarrow \mathbb{R}_+$ that satisfies the following properties

- $f(e) \le c(e)$ for all $e \in E$
- $\sum_{v':(v',v)\in E} f(v', v) = \sum_{v':(v,v')\in E} f(v, v')$ for all $v \in V \setminus \{s, t\}$.

The first properties requires that we cannot send more flow through an edge than its capacity, while the second makes sure that the flow is conserved throughout the network, i.e. that for every node its total incoming flow equals its total outgoing flow. Then, the *total flow* of G is sum of the flows leaving s (or arriving into t, since these two must be equal) $\sum_{v\in N(s)} f(s, v)$.

An *$(s-t)$ cut* of G is a bipartition $S \cup T = V$ of the node set such that $s \in S$ and $t \in T$. Some times we refer to set S as the left set and to T as the right one. Then the *value* or *capacity* of a cut (S, T) is simply the sum of the capacities of the edges going from S to T: $\sum_{v\in S, v'\in T:(v,v')\in E} c(v, v')$.

The well-known *max-flow min-cut* theorem states that the maximum total flow in a graph equals the minimum value of a cut of it. In particular, the capacity of every feasible cut is an upper bound to any feasible flow.

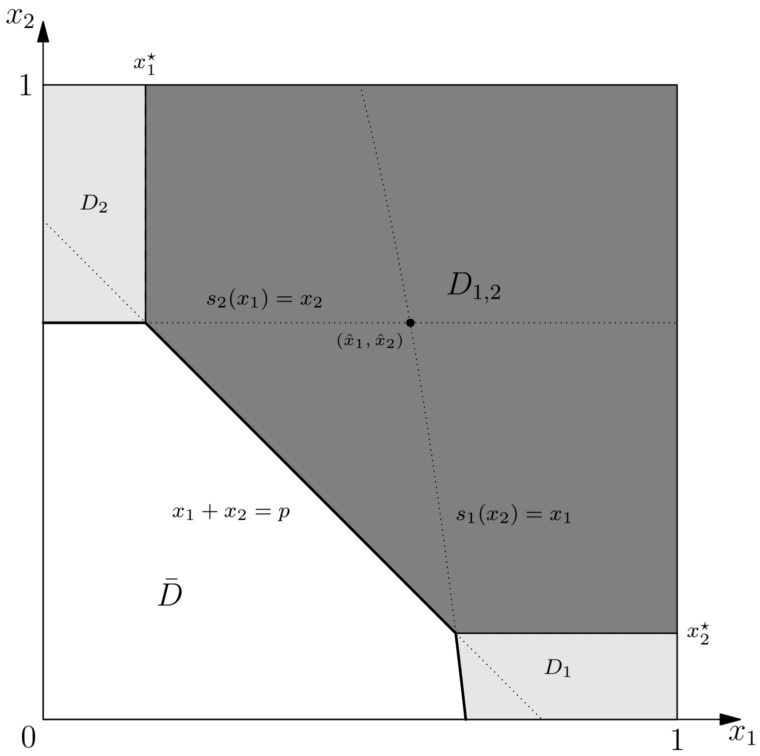


Figure 5.1: The type space partitioning of the optimal selling mechanism for two independent items, one following a uniform distribution and the other an exponential with parameter $\lambda = 1$. Here $s_1(t) = (2-t)/(3-t)$, $s_2(t) = 2 - W(2e) \approx 0.625$ and $p \approx 0.787$. In region D_1 (light grey) item 1 is sold deterministically and item 2 with a probability of $-s_1'(x_2)$, in D_2 (light grey) only item 2 is sold and region $D_{1,2}$ (dark grey) is where the full bundle is sold deterministically, for a price of p.

5.3.2 Partitioning of the Type Space

Due to the fact that the derivatives of functions s_j in Theorem 5.1 are above -1, each curve $x_1 = s_1(x_2)$ and $x_2 = s_2(x_1)$ can intersect the full-bundle line $x_1 + x_2 = p$ at most at a single point. So let $x_2^* = x_2^*(p), x_1^* = x_1^*(p)$ be the coordinates of these intersections, respectively, i.e. $s_1(x_2^*) = p - x_2^*$ and $s_2(x_1^*) = p - x_1^*$. If such an intersection does not exist, just define $x_2^* = 0$ or $x_1^* = 0$.

The construction and the optimal mechanism given in Theorem 5.1 then gives rise to the following partitioning of the type space I^2 (see Figure 5.1):

- Region $\bar{D} = I^2 \setminus D$ where no item is allocated
- Region $D_1 = \{\mathbf{x} \in I^2 \mid x_1 \geq s_1(x_2) \wedge x_2 \leq x_2^*\}$ where item 1 is sold with probability 1 and item 2 with probability $s_1'(x_2)$ for a price of $s_1(x_2) - x_2 s_1'(x_2)$
- Region $D_2 = \{\mathbf{x} \in I^2 \mid x_2 \geq s_2(x_1) \wedge x_1 \leq x_1^*\}$ where item 2 is sold with probability 1 and item 1 with probability $s_2'(x_1)$ for a price of $s_2(x_1) - x_1 s_2'(x_1)$
- Region $D_{1,2} = D \setminus D_1 \cup D_2 = \{\mathbf{x} \in I^2 \mid x_1 + x_2 \geq p \wedge x_1 \geq x_1^* \wedge x_2 \geq x_2^*\}$ where both items are sold deterministically in a full bundle of price p.

Under this decomposition, by (5.12):

$$\int_{D_1} h(\mathbf{x})\,dx_1\,dx_2 = \int_0^{x_2^*}\int_{s_1(x_2)}^1 h(\mathbf{x})\,dx_1\,dx_2 = f_1(1)F_2(x_2^*)$$

so expression (5.10) can be written equivalently as

$$\int_{D_{1,2}} h(\mathbf{x})\,dx_1\,dx_2 = f_1(1)(1-F_2(x_2^*)) + f_2(1)(1-F_1(x_1^*)). \tag{5.13}$$

Our approach into proving Theorem 5.1 will be to show the existence of a pair of dual solutions z_1, z_2 with respect to which the utility function u given by the theorem indeed satisfies complementarity. Notice here the existential character of our technique: our duality approach offers the advantage to use the proof of just the existence of such duals, without having to explicitly describe them and compute their objective value in order to prove optimality, i.e. that the primal and dual objectives are indeed equal. Also notice that the utility function u given by Theorem 5.1 is convex with nonnegative derivatives by construction, so in case someone shows optimality for u in the relaxed primal duality setting, then u must also be optimal among all feasible mechanisms.

Define function $W : I^2 \to \mathbb{R}_+$ by

$$W(\mathbf{x}) = \begin{cases} h(\mathbf{x}), & \text{if } \mathbf{x} \in D, \\ 0, & \text{otherwise,} \end{cases}$$

where D is defined in Section 5.3.2 (see Figure 5.1). If one could decompose W into functions $w_1, w_2 : I^2 \to \mathbb{R}_+$ such that

$$w_1(\mathbf{x}) + w_2(\mathbf{x}) = W(\mathbf{x}) \tag{5.14}$$

$$\int_0^1 w_j(\mathbf{x})\,dx_j = f_j(1)f_{-j}(x_{-j}), \qquad j = 1, 2, \tag{5.15}$$

for all $\mathbf{x} \in I$, and w_j is almost everywhere continuous with respect to its j-th coordinate, then by defining

$$z_j(\mathbf{x}) = \int_0^{x_j} w_j(t, x_{-j})\,dt$$

we'll have

$$\frac{\partial z_1(\mathbf{x})}{\partial x_1} + \frac{\partial z_2(\mathbf{x}_2)}{\partial x_2} = \begin{cases} h(\mathbf{x}), & \text{for } \mathbf{x} \in D, \\ 0, & \text{otherwise,} \end{cases} \tag{5.16}$$

$$z_j(0, x_{-j}) = 0, \qquad j = 1, 2, \tag{5.17}$$

$$z_j(1, x_{-j}) = f_j(1)f_{-j}(x_{-j}), \qquad j = 1, 2. \tag{5.18}$$

If the requirements of Theorem 5.1 hold, then it is fairly straightforward to get such a decomposition in certain regions. In particular, we can set $w_1 = w_2 = 0$ in $I^2 \setminus D$,

$w_1 = W = h$ and $w_2 = 0$ in D_1 and $w_2 = W = h$ and $w_1 = 0$ in D_2. Then, by (5.12), it is not difficult to see that indeed conditions (5.14)–(5.15) are satisfied. However, *it is highly non-trivial how to create such a matching in the remaining region* $D_{1,2}$ and that is what the proof of Lemma 5.3 achieves, with the assistance of the geometric Lemma 5.2, in the remaining of this section. This is the most technical part of the current chapter.

In any case, if we are able to get such a decomposition, by the previous discussion that would mean that functions $z_1, z_2 : I^2 \to \mathbb{R}_+$ are *feasible dual* solutions: it is trivial to verify that properties (5.16)–(5.18) satisfy the dual constraints (5.3)–(5.5). But most importantly, the *equalities* in properties (5.16)–(5.18) and the way w_1 and w_2 are defined in regions D_1 and D_2 tell us something more: that this pair of solutions would satisfy complementarity with respect to the primal given in (5.11) and whose allocation is analyzed in detail in Section 5.3.2, thus proving that this mechanism is optimal and thus establishing Theorem 5.1.

5.3.3 Deficiency

Now recall the notion of deficiency from Definition 4.3, adapted to our two-dimensional setting: for any body $S \subseteq I^2$ define its *deficiency* (with respect to distributions f_1, f_2) to be

$$\delta(S) \equiv \int_S h(\mathbf{x})\, d\mathbf{x} - f_2(1) \int_{S_1} f_1(x_1)\, dx_1 - f_1(1) \int_{S_2} f_2(x_2)\, dx_2,$$

where S_1, S_2 denote S's projections to the x_1 and x_2 axis, respectively. Then, the following analogue to Theorem 4.3 from the uniform setting of Chapter 4 can be shown, though with a different proof technique:

Lemma 5.2. *If the requirements of Theorem 5.1 hold, then no body $S \subseteq D_{1,2}$ has positive deficiency.*

Proof. To get to a contradiction, assume that there is body $S \subseteq D_{1,2}$ with $\delta(S) > 0$. First, we'll show that without loss S can be assumed to be upwards closed. Intuitively, we'll show that one can push mass of S to the right or upwards, without reducing its deficiency. By Assumption 5.1 function $h(\mathbf{x}) - f_2(1) f_1(x_1)$ is nonnegative. Then, if there exists a nonempty horizontal line segment $S|_{x_2:t}$ of S at some height $x_2 = t$, then we can assume that this line segment fills the entire available horizontal space of $D_{1,2}$: if that was not the case, and there existed a small interval $[\alpha, \beta] \times t$ that was not in S, then we could add it to it, not increasing the projection towards the x_2-axis (it is already covered by the other existing points at $x_2 = t$) and the projection towards the x_1-axis is increased at most by $\beta - \alpha$, leading to a change to the overall deficiency by at most $\int_\alpha^\beta h(\mathbf{x})\, dx_1 - f_2(1) \int_\alpha^\beta f_1(x_1)\, dx_1$, which is nonnegative[2].

[2]We must mention here that the assumption of the nonnegativity of $h(\mathbf{x}) - f_2(1) f_1(x_1)$ could be replaced by that of $h(\mathbf{x}) - f_2(1) f_1(x_1)$ being increasing with respect to x_1 and the argument would

So S can be assumed to be the intersection of $D_{1,2}$ with a box, i.e. $S = [t_1, 1] \times [t_2, 1] \cap D_{1,2}$, where $t_1 \geq x_1^*$ and $t_2 \geq x_2^*$. This also means that its projections are $S_1 = [t_1, 1]$ and $S_2 = [t_2, 1]$. Now consider the lowest horizontal slice $S|_{x_2:t_2}$ of S. It obviously lies within $D_{1,2}$. But from condition (5.12) so do all horizontal line segments of the form $[s_1(x_2), 1]$ for any $x_2 \in [x_2^*, t_2]$: $s_1(x_2)$ is decreasing and specifically less steeply than the line $-x_2 + p$ which is the boundary of $D_{1,2}$. So, by adding all these segments to S we won't increase the projections towards the x_1-axis (these are covered already by $S|_{x_2:t_2}$, which has to be a superset of $[s_1(t_2), 1]$, otherwise it would have a negative deficiency, see (5.12)) and the new projections towards the x_2-axis are dominated by the increase of the area of S (this segments have nonnegative deficiency). So, S can be assumed to project in the entire boundaries $[x_1^*, 1]$ and $[x_2^*, 1]$ of $D_{1,2}$ and thus, since h is nonnegative, S can be assumed to fill the entire $D_{1,2}$ region. But by the definition of price p in Theorem 5.1, $\delta(D_{1,2}) = 0$ which concludes the proof. □

5.3.4 Dual Solution and Optimality

Notice here that Theorem 5.1 ensures the existence of a full-bundling price in (5.10). This needs to be proven. Indeed, quantity $\int_D h(\mathbf{x})\, d\mathbf{x}$ continuously (weakly) increases as p decreases, and for $p = 0$

$$\begin{aligned}\int_D h(\mathbf{x})\, d\mathbf{x} &= \int_0^1 \int_0^1 3f_1(x_1)f_2(x_2) + x_1 f_1'(x_1)f_2(x_2) + y f_2'(x_2)f_1(x_1)\, dx_1\, dx_2 \\ &= 3 + (f_1(1) - 1) + (f_2(1) - 1) = 1 + f_1(1) + f_2(1) \\ &> f_1(1) + f_2(1)\end{aligned}$$

while for $p = \hat{x}_1 + \hat{x}_2$, where $(\hat{x}_1, \hat{x}_2)$ is the unique point of intersection of the curves $x_2 = s_1(x_1)$ and $x_1 = s_2(x_2)$ in I^2 (such a point certainly exists because s_1 and s_2 are defined over the entire I),

$$\begin{aligned}\int_{D_{1,2}} h(\mathbf{x})\, d\mathbf{x} &= \int_{\hat{x}_2}^1 \int_{\hat{x}_1}^1 h(\mathbf{x})\, d\mathbf{x} \\ &\leq \int_{\hat{x}_2}^1 \int_{s_1(x_2)}^1 h(\mathbf{x})\, d\mathbf{x} \\ &= \int_{\hat{x}_2}^1 f_1(1)f_2(x_2)\, dx_2 \\ &= f_1(1)(1 - F_2(\hat{x}_2)) \\ &\leq f_1(1)(1 - F_2(\hat{x}_2)) + f_2(1)(1 - F_1(\hat{x}_1)),\end{aligned}$$

still cary through: we can move entire columns of S to the right, pushing elements horizontally; the projection towards axis x_2 again remains unchanged, and because of the monotonicity of $h(\mathbf{x}) - f_2(1)f_1(x_1)$, the overall deficiency will not decrease since we are integrating over higher values of x_1.

This means that the monotonicity of $h(\mathbf{x}) - f_j(x_j)j_{-j}(1)$ with respect to x_j can replace its non-negativity in the initial Assumption 5.1 (while still maintaining the regularity requirement of $h(\mathbf{x})$ being nonnegative) without affecting the main results of this chapter, namely Theorems Theorems 5.1 to 5.3.

the first inequality holding because h is nonnegative and $s_1(x_2) \leq s_1(\hat{x}_2) = \hat{x}_1$ (s_1 is decreasing), and the second equality by substituting (5.12), and from (5.13) this means that $\int_D h \, d\mathbf{x} \leq f_1(1) + f_2(1)$.

Combining the above, indeed there must be a $p \in [0, \hat{x}_1 + \hat{x}_2]$ such that $\int_D h \, d\mathbf{x} = f_1(1) + f_2(1)$. In fact, using this argument, if for $p = \min \{s_1(0), s_2(0)\}$ it is $\int_D h \, d\mathbf{x} < f_1(1) + f_2(1)$ then p must go below this value to get a solution, meaning that the full-bundling region will cover the rest of the regions D_1 and D_2, i.e. $D = D_{1,2}$, and the mechanism defined by (5.11) is a deterministic full-bundling.

The following lemma will complete the proof of Theorem 5.1. It is the most technical part of this chapter, and utilizes a max-flow min-cut argument in order to prove the existence of a feasible dual pair z_1, z_2 that satisfies the complementarity conditions with respect to the utility function given by Theorem 5.1, thus establishing optimality. It is inspired by the bipartite matching approach in Chapter 4 where Hall's theorem is used in order to prove existence, in the special case of uniformly distributed items. Here we need to abstract and generalize our approach in order to incorporate general distributions in the most smooth way possible. The proof has a strong geometric flavour, which is achieved by utilizing the notion of deficiency that was introduced in Section 5.3.3 and using Lemma 5.2.

Lemma 5.3. *Assume that the conditions of Theorem 5.1 hold. Then for arbitrary small $\varepsilon > 0$, there exist feasible dual solutions z_1, z_2 which are ε-complementary to the (primal) u given by (5.11). Therefore, the mechanism induced by u is optimal.*

Proof. Following the discussion in Section 5.2, we would like to decompose W into the desired functions w_1 and w_2 within $D_{1,2}$, i.e. such that they satisfy (5.14)–(5.15). In fact, we are aiming for ε-complementarity, so we can relax conditions (5.15) a bit:

$$\int_0^1 w_j(\mathbf{x}) \, dx_j \leq f_j(1) f_{-j}(x_{-j}) + \varepsilon' \tag{5.19}$$

To be precise, the ε-complementarity of Lemma 3.2 dictates that regarding these conditions we must show that for a.e. $\mathbf{x} \in D_{1,2}$ property (5.7) holds (conditions (5.6) and (5.8) are immediately satisfied with strong equality, by (5.16) and the fact that within $D_{1,2}$ both items are sold deterministically with probability 1.). But since $u(\mathbf{x}) \leq x_1 + x_2 \leq 2$ for all $x_1, x_2 \in I$ (u's derivatives are at most 1 with respect to any direction) and by Assumption 5.0 exists $M > 0$ such that $f_1(1) f_2(x_2), f_2(1) f_1(x_1) \geq M$ for all $\mathbf{x} \in D_{1,2}$, indeed (5.19) is enough to guarantee ε complementarity if one ensures $\varepsilon' \leq \varepsilon M/2$. So, the remaining of the proof is dedicated into constructing nonnegative, a.e. continuous functions w_1 and w_2 over $D_{1,2}$, such that $w_1 + w_2 = h$ and (5.19) are satisfied.

We will do that by constructing an appropriate graph and recovering w_1 and w_2 as "flows" through its nodes, deploying the min-cut max-flow theorem to prove existence. To start, we pick an arbitrary small $\delta > 0$ and discretize I^2 into a lattice of δ-size boxes $[(i-1)\delta, i\delta] \times [(j-1)\delta, j\delta]$, where $i, j = 1, 2, \ldots, 1/\delta$, selecting δ such that $1/\delta$

is an integer. Denote the intersection of such a box with $D_{1,2}$ by $B_{i,j}$. Also, let B_i^1 denote the projection of all nonempty $B_{i,j}$'s, as j ranges, towards the x_1-axis and B_j^2 towards the x_2-axis, as i ranges. Note that these are well-defined in this way, since by the geometry of region $D_{1,2}$ two nonempty $B_{i,j}$, $B_{i',j'}$ will have the same vertical projection if $i = i'$ and the same horizontal if $j = j'$. Also, it is a simple fact to observe that all B_i^1 and B_j^2 are single-dimensional real intervals of length at most δ.

Now let us construct a directed graph $G = (V, E)$, together with a capacity function $c(e)$ for all edges $e \in E$. Initially, for any pair (i, j) such that $B_{i,j}$ has positive (two-dimensional Lebesgue) measure we insert a node $v(i, j)$ in V. We'll call these nodes *internal* and we'll denote them by V_o. Also, for any internal node $v(i, j)$ we add nodes $v_1(i)$ and $v_2(j)$ corresponding to entire columns and rows, calling them *column* and *row* vertices and denoting them by V_1 and V_2, respectively. Finally there are two special nodes, a source σ and a destination τ. From the source to all internal nodes $v = v(i, j)$ we add an edge (σ, v) with capacity equal to the area of $B_{i,j}$ under h, i.e. $c(\sigma, v) = \int_{B_{i,j}} h(\mathbf{x})\, d\mathbf{x}$. From any internal node $v = v(i, j)$ to its external column and row nodes $v_1 = v_1(i)$ and $v_2 = v_2(j)$ we add edges with capacities $c(v, v_1) = c(v, v_2) = c(\sigma, v)$ equal to the internal node's incoming edge capacity from the source. Finally, for all external nodes $v_1(i) \in V_1$ and $v_2(j) \in V_2$ we add edges towards the destination τ with capacities $c(v_1, \tau) = f_2(1) \int_{B_i^1} f_1(x_1)\, dx_1$ and $c(v_2, \tau) = f_1(1) \int_{B_j^2} f_2(x_2)\, dx_2$, respectively. The structure of graph G is depicted in Figure 5.2.

As a first observation, notice that the maximum flow that can be sent from σ within the graph is $\int_{D_{1,2}} h(\mathbf{x})\, dx\, dy$ and the maximum flow that τ can receive is

$$f_2(1) \int_{x_1^*}^{1} f_1(x_1)\, dx_1 + f_1(1) \int_{x_2^*}^{1} f_2(x_2)\, dx_2$$

(remember that the projection of $D_{1,2}$ to the x_1-axis is $[x_1^*, 1]$ and to the x_2-axis $[x_2^*, 1]$). But, from the way the entire region D is constructed, we know that the above two quantities are equal (see (5.13)). Let's denote this value by ψ. Next, we will prove that indeed one can create a feasible flow through G that achieves that maximum value ψ. From the max-flow min-cut theorem, it is enough to show that the minimum (σ, τ)-cut of G has a value of at least ψ. To do that, we'll show that $(\sigma, V \setminus \{\sigma\})$ is a minimum cut of G.

Indeed, let $(S, V \setminus S)$ be a (σ, τ)-cut of G. First, let there be an edge (v, v_j) crossing the cut, i.e. $v \in S$ and $v_j \notin S$, with v internal node and v_j external. Then, by moving v at the other side of the cut, i.e. removing it from S, we would create at most a new edge contributing to the cut, namely (σ, v) but also destroy at least one edge (v, v_j). Since the capacities of these two edges are the same, the overall effect would be to get a new cut with weakly smaller value. So, from now on we can assume that for all edges (v, v_j) of G, if $v \in S$ then also $v_j \in S$. Under this assumption, if $S_o = V_o \cap S$ denotes the set of internal nodes belonging at the left side of the cut, for every $v \in S_o$ all edges

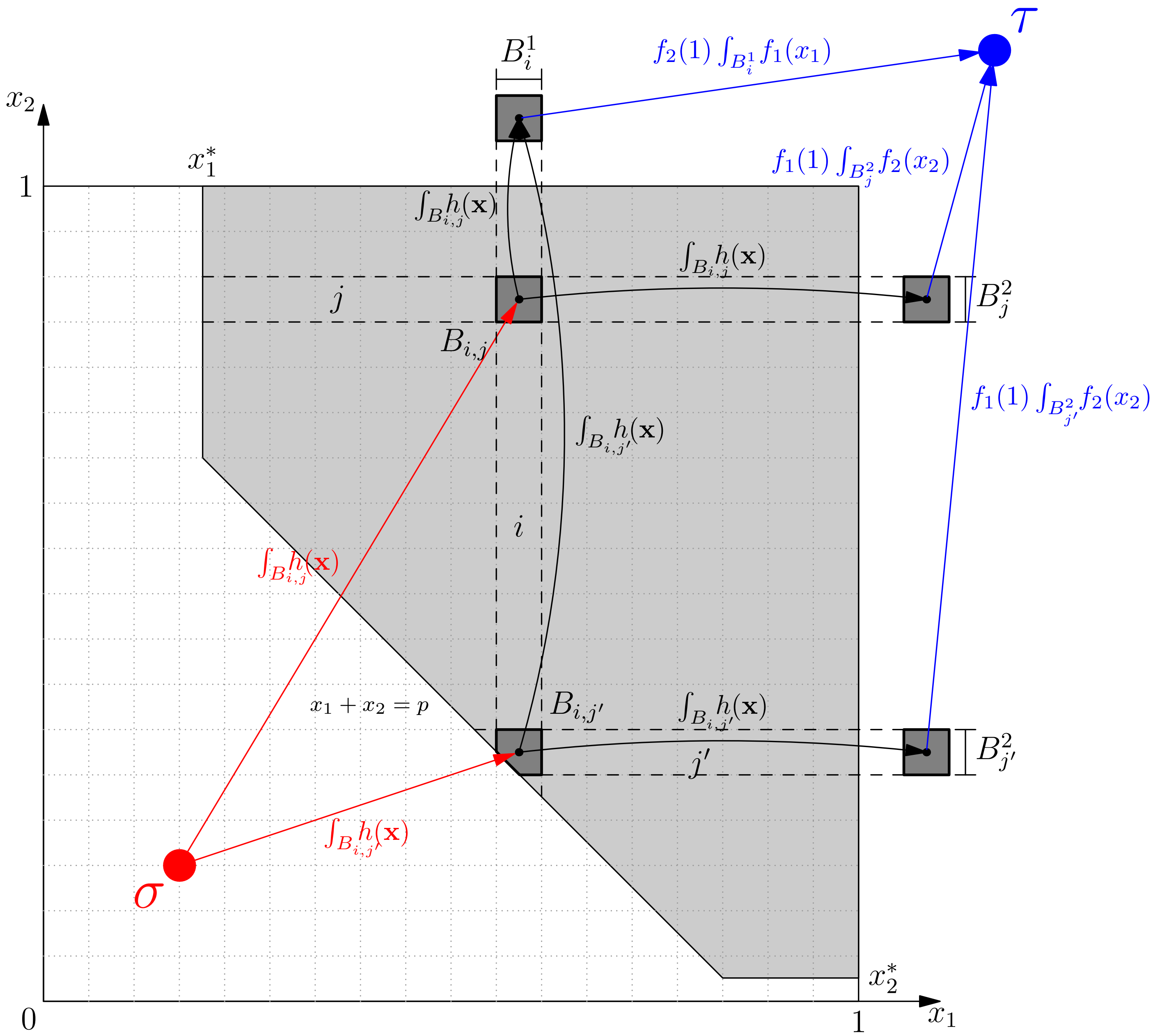


Figure 5.2: The graph G in the proof of Lemma 5.3. Every internal node $B_{i,j}$ of region $D_{1,2}$ can receive at most $\int_{B_{i,j}} h(\mathbf{x})\,d\mathbf{x}$ flow from the source node σ and can send at most that amount to each one of its neighbouring external nodes B_i^1 and B_j^2. Every external node B_i^1 and B_j^2 is connected to the destination τ with edges of capacity $f_2(1)\int_{B_i^1} f_1(x_1)\,dx_1$ and $f_1(1)\int_{B_j^2} f_2(x_2)\,dx_2$, respectively. Internal $B_{i,j}$'s are two-dimensional intersections of δ-boxes with $D_{1,2}$, while the external ones, B_i^1 and B_j^2 are single dimensional intervals of length δ.

(v, v_j) adjacent to v will not cross the cut. However, this means that all edges (v_j, τ), where $v_j \in N(v)$[3], do contribute to the cut. But then, if we remove all nodes in S_o, together with their neighbouring external nodes $N(S_o)$ at the other side of the cut, we increase the cut's value by at most $\sum_{v \in S_o} c(\sigma, v)$ and at the same time reduce it by at least $\sum_{v_j \in N(S_o)} c(v_j, \tau)$. However, by the way graph G is constructed, this corresponds to an overall increase in the cut of at least

$$\int_B h(\mathbf{x})\, d\mathbf{x} - f_2(1) \int_{B_1} f_1(x_1)\, dx_1 - f_1(1) \int_{B_2} f_2(x_2)\, dx_2,$$

where $B = \cup_{v(i,j) \in S_o} B_{i,j}$ is the region of $D_{1,2}$ covered by the boxes of nodes in S_o and B_1, B_2 are the projections of this body to the horizontal and vertical axis, respectively. From Lemma 5.2 this difference must be nonpositive, thus this change results in a cut of an even (weakly) smaller value. The above arguments show that indeed the cut that has only σ remaining at its left side is a minimum one.

So, there must be a flow $\phi : E \longrightarrow \mathbb{R}_+$, achieving to transfer a total value of ψ through G (see Section 5.3.1). As we argued above though, by the construction of G, to achieve this value of ψ, the full capacity of *all* edges (σ, v) as well as that of all (v_j, τ) must be used. So, this flow f manages to elegantly separate all incoming flow $\phi(\sigma, v(i, j)) = \int_{B_{i,j}} h(\mathbf{x})\, d\mathbf{x}$ towards an internal box of $D_{1,2}$, into a sum of flows $\phi(v(i, j), v_1(i)) + \phi(v(i, j), v_2(j))$ towards its external neighbours. But this is exactly what we need in order to construct our feasible dual solution! For simplicity, denote this incoming flow $\phi(i, j)$ and the outgoing ones $\phi_1(i, j)$ and $\phi_2(i, j)$, respectively. Then, define the functions w_1, w_2 throughout $D_{1,2}$ by

$$w_1(\mathbf{x}) = \frac{\phi_1(i, j)}{\phi(i, j)} h(\mathbf{x}) \qquad \text{and} \qquad w_2(\mathbf{x}) = \frac{\phi_2(i, j)}{\phi(i, j)} h(\mathbf{x}),$$

where $B_{i,j}$ is the discretization box where point $\mathbf{x}$ of $D_{1,2}$ belongs to. In that way, first notice that we achieve $w_1 + w_2 = h$. Secondly, functions w_1 and w_2 are almost everywhere continuous, since the values of the flows are constant within the boxes, and our discretization is finite. The only remaining property to prove is (5.19).

Fix some height $x_2 = \tilde{x}_2$ such that this horizontal line intersects $D_{1,2}$. We'll prove that

$$\int_0^1 w_1(x_1, \tilde{x}_2)\, dx_1 - f_1(1) f_2(\tilde{x}_2) \leq \varepsilon'.$$

Value $\tilde{x}_2$ falls within some interval of the discretization, let $\tilde{x}_2 \in [(\tilde{j} - 1)\delta, \tilde{j}\delta] = B^2_{\tilde{j}}$. The average value of function $f_1(1) f_2(x_2)$ (with respect to x_2) within this interval is

$$\frac{1}{\delta} f_1(1) \int_{B^2_{\tilde{j}}} f_2(x_2)\, dx_2 = c(v_2(\tilde{j}), \tau)/\delta$$

[3]Recall that $N(v)$ denotes the set of neighbours of v in graph G.

and the average value of $\int_0^1 w_1(\mathbf{x})\,dx_1$ is

$$\frac{1}{\delta}\int_{B^2_{\tilde{j}}}\int_0^1 w_1(\mathbf{x})\,dx_1 = \frac{1}{\delta}\sum_i \int_{B_{i,\tilde{j}}} w_1(\mathbf{x})\,d\mathbf{x} = \frac{1}{\delta}\sum_i \frac{\phi_1(i,j)}{\phi(i,j)}\int_{B_{i,\tilde{j}}} h(\mathbf{x})\,d\mathbf{x} = \sum_i \phi_1(i,\tilde{j})/\delta.$$

But since the sum of the outgoing flows over any horizontal line of internal nodes of the graph (here $j = \tilde{j}$) must equal the outgoing flow of the corresponding external node (here $v_2(\tilde{j})$), the above quantities are equal. Thus, by selecting the discretization parameter δ small enough, we can indeed make the values $\int_0^1 w_1(x_1, \tilde{x}_2)\,dx_1$ and $f_1(1)f_2(\tilde{x}_2)$ to be ε' close to each other. This should feel intuitively clear, and it relies on the uniform continuity of functions f_2 and h, but we also give a formal proof below:

Functions f_2 and $\int_0^1 w_1(x_1, \tilde{x}_2)\,dx_1$ are continuous in the interval $B^2_{\tilde{j}}$, so by the Mean Value Theorem there exist $\bar{x}_2, \bar{\bar{x}}_2 \in B^2_{\tilde{j}}$ such that

$$\int_0^1 w_1(x_1, \bar{x}_2)\,dx = \frac{1}{\delta}\int_{B^2_{\tilde{j}}}\int_0^1 w_1(\mathbf{x})\,d\mathbf{x} = \frac{1}{\delta} f_1(1)\int_{B^2_{\tilde{j}}} f_2(x_2)\,dx_2 = f_1(1)f_2(\bar{\bar{x}}_2) \tag{5.20}$$

Notice that both $\bar{x}_2$ and $\bar{\bar{x}}_2$ are δ-close to $\tilde{x}_2$. Function f_2 is uniformly continuous, so one can pick δ small enough in order to

$$f_1(1)f_2(\bar{\bar{x}}_2) - f_1(1)f_2(\tilde{x}_2) \le \varepsilon'/2. \tag{5.21}$$

In the same way, because h is uniformly continuous, we can select δ small enough so that $h(x_1, \bar{x}_2) - h(x_1, \tilde{x}_2) \le \varepsilon'/3$ for all $x_1 \in I$, and that would give

$$\begin{aligned}\left|\int_0^1 w_1(x_1, \bar{x}_2)\,dx_1 - \int_0^1 w_1(x_1, \tilde{x}_2)\,dx_1\right| &\le \sum_i \frac{f_1(i,j)}{f(i,j)}\int_{B^1_i} \left|h(x_1, \bar{x}_2) - h(x_1, \tilde{x}_2)\right|\,dx_1 \\ &\quad + |\bar{x}_2 - \tilde{x}_2|\,\|h\|_\infty \\ &\le \sum_i \int_{B^1_i} \left|h(x_1, \bar{x}_2) - h(x_1, \tilde{x}_2)\right|\,dx_1 + \delta\,\|h\|_\infty \\ &\le \int_0^1 \frac{\varepsilon'}{3}\,dx_1 + \delta\,\|h\|_\infty \\ &\le \varepsilon'/2, \end{aligned} \tag{5.22}$$

for choosing a small enough value for δ, since $\|h\|_\infty \equiv \sup_{\mathbf{x}\in I^2} h(\mathbf{x})$ is a fixed constant (because h is continuous). The last additive term in the first inequality accounts for the fact that the length of the intersections of horizontal lines $x_2 = \bar{x}_2$ and $x_2 = \tilde{x}_2$ with $D_{1,2}$ may differ by $|\bar{x}_2 - \tilde{x}_2|$ (remember that the boundary of $D_{1,2}$ is 45°–line).

Finally, by plugging in inequalities (5.21) and (5.22) into (5.20) we get the desired

$$\left|\int_0^1 w_1(x_1, \tilde{x}_2)\,dx_1 - f_1(1)f_2(\tilde{x}_2)\right| \le \varepsilon'.$$

□

5.4 Optimal Mechanisms

5.4.1 The Case of Identical Items

In this section we focus on the case of identically distributed valuations, i.e. $f_1(t) = f_2(t) \equiv f(t)$ for all $t \in I$, and we provide clear and simple conditions under which the critical property (5.9) of Theorem 5.1 hold.

First notice that in this case the regularity Assumption 5.1 gives $3 + \frac{x_1 f'(x_1)}{f(x_1)} + \frac{x_2 f'(x_2)}{f(x_2)} \geq 0$ a.e. in I^2 (since f is positive) and thus $\frac{tf'(t)}{f(t)} \geq -\frac{3}{2}$ for a.e. $t \in I$. An equivalent way of writing this is that $t^{3/2} f(t)$ is increasing, which interestingly is the complementary case of that studied by Hart and Nisan [38] for two i.i.d. items: they show that when $t^{3/2} f(t)$ is decreasing, then deterministically selling in a full bundle is optimal.

Theorem 5.2. *Assume that $G(t) = tf(t)/(1 - F(t))$ and $H(t) = tf'(t)/f(t)$ give rise to well defined, differentiable functions over I, G being strictly increasing and convex, H decreasing and concave, with $G + H$ increasing and $G(1) \geq 2 + H(0)$. Then the requirements of Theorem 5.1 are satisfied. In particular*

$$s(t) = G^{-1}(2 + H(t))$$

and, if

$$\int_0^1 \int_0^1 h(\mathbf{x})\, d\mathbf{x} - \int_0^p \int_0^{p-x_2} h(\mathbf{x})\, d\mathbf{x} - 2f(1) \tag{5.23}$$

is nonpositive for $p = s(0)$ then the optimal selling mechanism is the one offering deterministically the full bundle for a price of p being the root of (5.23) in $[0, s(0)]$, otherwise the optimal mechanism is the one defined by the utility function

$$u(\mathbf{x}) = \max\{0, x_1 - s(x_2), x_2 - s(x_1), x_1 + x_2 - p\}$$

with $p = x^* + s(x^*)$, where $x^* \in [0, s(0)]$ is the constant we get by solving

$$\int_{x^*}^{s(x^*)} \int_{s(x^*)+x^*-x_2}^{1} h(\mathbf{x})\, d\mathbf{x} + \int_{s(x^*)}^{1} \int_{x^*}^{1} h(\mathbf{x})\, d\mathbf{x} = 2f(1)(1 - F(x^*)). \tag{5.24}$$

Proof. Function G is strictly monotone, thus invertible and has a range of $[G(0), G(1)] = [0, G(1)] \supseteq [0, 2 + H(0)]$. By Assumption 5.1 and the previous discussion, it must be $tf'(t)/f(t) \geq -3/2$, so $2 + H(t) \geq 1/2 > 0$ for all $t \in I$. Thus, $s(t) = G^{-1}(2 + H(t))$ is well defined and furthermore it is decreasing, since G is increasing and H decreasing. Also, by the way s is defined we get that for all t: $G(s(t)) = 2 + H(t)$, which is exactly condition (5.9) of Theorem 5.1.

It remains to be shown that s is concave and that $s'(t) > -1$. From the definition of s, $s'(t) = H'(t)/G'(s(t))$. Function H is decreasing and concave, so $H'(t)$ is negative and decreasing, and function G is increasing and convex and s decreasing, so $G'(s(t))$ is positive and decreasing. Combining these we get that the ratio $H'(t)/G'(s(t))$ is decreasing, proving that s is concave. Finally, notice that since we are in a two item i.i.d. setting, the only part of curve $x_2 = s(x_1)$ that matters and may appear in the utility of the resulting mechanism (5.11) is the one where $x_1 \leq x_2$ (curves $x_2 = s(x_1)$ and $x_1 = s(x_2)$ will intersect on the line $x_1 = x_2$), so we only have to show that $s'(t) > -1$ for $t \leq s(t)$. Indeed, in that case $G'(t) \leq G'(s(t))$, so $s'(t) = H'(t)/G'(s(t)) \geq H'(t)/G'(t)$ and thus it is enough to show that $H'(t) - G'(t) \geq 0$ which we know holds since $H + G$ is assumed to be increasing.

□

Corollary 5.1 (Monomial Distributions)**.** *The optimal selling mechanism for two i.i.d. items with valuations from the family of distributions with densities* $f(t) = (c+1)t^c$, $c \geq 0$, *is deterministic. In particular, it offers each item for a price of*

$$s = \sqrt[c+1]{\frac{c+2}{2c+3}}$$

and the full bundle for a price of $p = s + x^*$, *where* x^* *is the solution to* (5.24).

Proof. For two monomial i.i.d. items with $f_1(t) = f_2(t) = (c+1)t^c$ we have $h(\mathbf{x}) = (c+1)^2(2c+3)x_1^c x_2^c \geq 0$, thus $h(\mathbf{x}) - f_2(1)f_1(x_1) = (c+1)^2 x_1^c\left((2c+3)x_2^c - 1\right)$ which is nonnegative for all $x_2 \geq \sqrt[c]{1/(2c+3)} \equiv \omega$. So, in order to make sure that Assumption 5.1 is satisfied, it is enough to show that $x^* \geq \omega$ because then $D_{1,2} \subseteq [\omega, 1]^2$. We'll soon show that this is indeed satisfied for all $c \geq 0$.

Applying Theorem 5.2 we compute: $G(t) = \frac{(c+1)t^{c+1}}{1-t^{c+1}}$ which is strictly increasing and convex in I and $H(t) = c$ which is constant and thus decreasing and concave. Also, it is trivial to deduce that $G+H$ is increasing and $\lim_{t\to 1^-} G(t) = \infty > 2+c = 2+H(0)$. Then, it is valid to compute $G^{-1}(t) = \left(\frac{3+2c}{2+c}\right)^{-\frac{1}{1+c}}$ and thus $s(t) = \sqrt[c+1]{\frac{c+2}{2c+3}}$ which is constant.

Regarding the computation of the full-bundle price p, condition (5.24) gives rise to quantity

$$\int_{x^*}^{s}\int_{s+x^*-x_2}^{1} x_1^c x_2^c \, d\mathbf{x} + \int_{s}^{1}\int_{x^*}^{1} x_1^c x_2^c \, d\mathbf{x} - \frac{2}{c+1}(1 - x^{*c+1}),$$

which by plugging-in $x^* = \omega$ and using the values of s and ω (as functions of c) one can see that it is positive for all $c \geq 0$. So, by the discussion in the beginning of Section 5.3.4 it can be deduced that the solution to (5.24) will be such that $x^* > \omega$. □

Notice that for $c = 0$ the setting of Corollary 5.1 reduces to a two uniformly distributed goods setting, and gives the well-known results of $s = 2/3$ and $p = (4 - \sqrt{2})/3$ (see e.g. [53]). For the linear distribution $f(t) = 2t$, where $c = 1$, we get $s = \sqrt{3/5}$ and $p \approx 1.091$.

Corollary 5.2 (Exponential Distributions)**.** *The optimal selling mechanism for two i.i.d. items with valuations exponentially distributed over I, i.e. having densities $f(t) = \frac{\lambda}{1-e^{-\lambda}} e^{-\lambda t}$, with $0 < \lambda \leq 1$, is the one having*

$$s(t) = \frac{1}{\lambda}\left[2 - \lambda t - W\left(e^{2-\lambda-\lambda t}(2-\lambda t)\right)\right]$$

and a price of $p = x^* + s(x^*)$ for the full bundle, where x^* is the solution to (5.24). Here W is Lambert's product logarithm function[4]*.*

Proof. For two i.i.d. exponentially distributed items with $f_1(t) = f_2(t) = \frac{\lambda}{1-e^{-\lambda}} e^{-\lambda t}$ we have

$$\begin{aligned} h(\mathbf{x}) - f_2(1) f_1(x_1) &= \frac{\lambda^2}{(e^{\lambda}-1)^2} e^{2-\lambda(x_1+x_2)} (3 - \lambda(x_1+x_2) - e^{\lambda x_2}) \\ &\geq \frac{\lambda^2}{(e^{\lambda}-1)^2} e^{2-\lambda(x_1+x_2)} (2 - \lambda(x_1+x_2)) \\ &\geq 0 \end{aligned}$$

for all $x_1, x_2 \in I$, since $\lambda \leq 1$.

Applying Theorem 5.2 we compute: $G(t) = \frac{\lambda t}{1-e^{-\lambda(1-t)}}$ which is strictly increasing and convex in I and $H(t) = -\lambda t$ which is decreasing and concave. Also, $G(t) + H(t) = \frac{\lambda t e^{-\lambda(1-t)}}{1-e^{-\lambda(1-t)}}$ is increasing and $\lim_{t \to 1^-} G(t) = \infty > 2 = 2 + H(0)$. Then, it is valid to compute $G^{-1}(t) = \frac{t - W\left(te^{t-\lambda}\right)}{\lambda}$ and thus $s(t) = \frac{1}{\lambda}\left[2 - \lambda t - W\left(e^{2-\lambda-\lambda t}(2-\lambda t)\right)\right]$. □

For example, for $\lambda = 1$ we get $s(t) = 2 - t - W\left(e^{1-t}(2-t)\right)$ and $p \approx 0.714$. Interestingly, to our knowledge this is the first example for an i.i.d. setting with valuations coming from a regular, continuous distribution over an interval $[0, b]$, where an optimal selling mechanism is *not* deterministic. Also notice how this case of exponential i.i.d. items on a bounded interval is different from the one on $[0, \infty)$: by [25, 30] we know that at the unbounded case the optimal selling mechanism for two exponential i.i.d. items is simply the deterministic full-bundling, but in our case of the bounded I this is not the case any more.

5.4.2 Non-Identical Items

An interesting aspect of the technique of Theorem 5.2 is that it can readily be used also for non identically distributed valuations. One just has to define $G_j(t) \equiv t f_j(t)/(1 - F_j(t))$ and $H_j(t) = t f_j'(t)/f_j(t)$ for both items $j = 1, 2$ and check again whether G_1, G_2 are strictly increasing and convex and H_1, H_2 nonnegative, decreasing and concave. Then, we can get $s_j(t) = G_j^{-1}(2 + H_{-j}(t))$ and check if $s_j(1) > -1$ and the price p of the full bundle can be given by (5.10). Again, a quick check of whether full

[4] Function W can be defined as the solution to $W(t)e^{W(t)} = t$.

bundling is optimal is to see if for $p = \min\{s_1(0), s_2(0)\}$ expression $\int_0^1 \int_0^1 h(\mathbf{x})\, d\mathbf{x} - \int_0^p \int_0^{p-x_2} h(\mathbf{x})\, d\mathbf{x} - f_1(1) - f_2(1)$ is nonpositive.

Example 1. Consider two independent items, one having uniform valuation $f_1(t) = 1$ and one exponential $f_2(t) = \frac{e^{-t}}{1-e^{-1}}$. Then we get that $s_1(t) = \frac{2-t}{3-t}$, $s_2(t) = 2 - W(2e) \approx 0.625$ and $p \approx 0.787$. The optimal selling mechanism offers either only item 2 for a price of $s_2 \approx 0.625$, or item 1 deterministically and item 2 with a probability $s_1'(x_2)$ for a price of $s_1(x_2) - x_2 s_1'(x_2)$, or the full bundle for a price of $p \approx 0.787$. You can see the allocation space of this mechanism in Figure 5.1.

5.5 Almost Optimal Auctions

In the previous sections we developed tools that, under certain assumptions, can give a complete closed-form description of the optimal selling mechanism. However, remember that the initial primal-dual formulation upon which our analysis was based, assumes a relaxed optimization problem. Namely, we dropped the convexity assumption of the utility function u. In the results of the previous sections this comes for free: the optimal solution to the relaxed program turns out to be convex anyway, as a result of the requirements of Theorem 5.1. But what happens if that was not the case? The following tool shows that even in that case our results are still applicable and very useful for both finding good upper bounds on the optimal revenue (Theorem 5.3) as well as designing almost-optimal mechanisms that have provably very good performance guarantees (Section 5.5.1).

Theorem 5.3. *Assume that all conditions of Theorem 5.1 are satisfied, except from the concavity of functions s_1, s_2. Then, the function u given by that theorem might not be convex any more and thus not a valid utility function, but it generates an* upper bound *to the optimal revenue, i.e.*

$$\textsc{Rev}(F_1 \times F_2) \leq \mathcal{R}(u; F_1 \times F_2).$$

In particular, this is the case if all the requirements of Theorem 5.2 hold except the concavity of H.

Proof. The proof is a straightforward result of the duality framework (see Section 3.1.2): By dropping only the concavity requirement of functions s_1 and s_2 but satisfying all the remaining conditions of Theorem 5.1, we still construct an optimal solution to the pair of primal-dual programs, meaning that function u produced in (5.11) maximizes $\mathcal{R}(u; F_1 \times F_2)$ over the space of all functions $u : I^2 \longrightarrow \mathbb{R}_+$ with partial derivatives in $[0, 1]$ (see (5.2)); the only difference is that u might not be convex since s_1, s_2 might not be concave any more. The actual optimal revenue objective $\textsc{Rev}(F_1 \times F_2)$ has the extra constraint of u being convex, thus, given that it is a maximization problem, it has to be that $\textsc{Rev}(F_1 \times F_2) \le \mathcal{R}(u; F_1 \times F_2)$. Finally, it is easy to verify in the proof ofTheorem 5.2 that dropping just the concavity requirement for H can only affect the concavity of functions s_1, s_2. □

Example 2 (Power-Law Distributions). A class of important distributions that falls into the description of Theorem 5.3 are the power-law distributions with parameters $0 < \alpha \le 2$. More specifically, these are the distributions having densities $f(t) = c/(t+1)^\alpha$, with the normalization factor c selected so that $\int_0^1 f(t)\, dt = 1$, i.e. $c = (a-1)/(1-2^{1-\alpha})$. It is not difficult to verify that these distributions satisfy Assumption 5.1. For example, for $\alpha = 2$ one gets $f(x) = 2/(x+1)^2$, the *equal revenue* distribution shifted in the unit interval. For this we can compute via Theorem 5.3 that $s(t) = \frac{1}{2}\sqrt{5 + 2t + t^2} - \frac{1}{2}(1+t)$ and $p \approx 0.665$, which gives an upper bound of $\mathcal{R}_{f,f}(u) \approx 0.383$ to the optimal revenue $\textsc{Rev}(F \times F)$.

5.5.1 Convexification

The approximation results described in Theorem 5.3 can be used not only for giving upper bounds on the optimal revenue, but also as a *design* technique for good selling mechanisms. Since the only deviation from a feasible utility function is the fact that function s is not concave (and thus u is not convex), why don't we try to "convexify" u, by replacing s by a concave function $\tilde{s}$, resulting in a new, convex and thus feasible auction $\tilde{u}$? If $\tilde{u}$ is "close enough" to the original u, by the previous discussion this would also result in good approximation ratios for the new, feasible selling mechanism.

More formally, the goal is to find a convex utility function $\tilde{u}$ such that

$$\frac{\mathcal{R}(u; F_1 \times F_2)}{\mathcal{R}(\tilde{u}; F_1 \times F_2)} \tag{5.25}$$

is minimized. Recall that this ratio is always above 1 and it is an upper bound on the approximation ratio $\frac{\textsc{Rev}(F_1 \times F_2)}{\mathcal{R}(\tilde{u}; F_1 \times F_2)}$ of the new feasible mechanism $\tilde{u}$, since by Theorem 5.3,

$$\mathcal{R}(\tilde{u}; F_1 \times F_2) \le \textsc{Rev}(F_1 \times F_2) \le \mathcal{R}(u; F_1 \times F_2).$$

Let's demonstrate this by an example, using the equal revenue distribution $f(t) = 2/(t+1)^2$ of the previous example. We need to replace s with a concave $\tilde{s}$ in the interval $[0, x^*]$. So let us choose $\tilde{s}$ to be the concave hull of s, i.e. the minimum concave function that dominates s. Since s is convex, this is simply the line that connects the two ends of the graph of s in $[0, x^*]$, that is, the line

$$\tilde{s}(t) = \frac{s(0) - s(x^*)}{x^*}(x^* - t) + s(x^*).$$

A calculation of the ratio (5.25) for this example of the equal revenue distribution gives a value of $1+3\times10^{-9}$, rendering the resulting new *valid* mechanism essentially optimal.

5.6 More on Uniform Distributions

In this section we turn our attention again to uniform distributions, and we deal with the problem of maximizing the expected revenue of a two-good monopolist when facing an additive buyer whose values for the goods come uniformly i.i.d. over general unit-length intervals $[c, c+1]$, $c > 0$. This problem was solved by Pavlov [66]. In the case of $c = 0$, the optimal selling mechanism is deterministic with prices $2/3$ for each of the items and $(4 - \sqrt{2})/3$ for their bundle. This result was already known by the work of Manelli and Vincent [53], and an alternative proof based on duality and complementarity can be found also in [31]. For $c \geq 0.077$, the optimal mechanism is again deterministic and it only offers the full bundle for a price of $(4c + \sqrt{4c^2 + 6})/3$. For the range in between, that is for $c \in (0, 0.077)$, Pavlov numerically computes that the optimal mechanism is a randomized one, with a menu-complexity [39] of 4.

Here, we present a very simple alternative proof for the cases of $c = 0$ and $c \geq 0.092$. For the remaining case, although we give the optimal solution to the primal-dual formulation, it turns out that it is *not* convex and that its objective value is strictly greater than the optimal revenue that can be achieved by any feasible selling mechanism. This is because our primal program is a relaxed version of the original revenue-maximization one, dropping the convexity constraint for the bidder's utility function. This relaxation has been the standard approach so far in duality theory frameworks for such problems (see [25, 31]). So, *this is a demonstration of the necessity, in general, of the convexity requirement for exact optimal mechanism design, even in the case of two regularly i.i.d. items*[5]. Nevertheless, on the positive side, we are able to demonstrate that the two solutions are practically very close to each-other (within a factor of 7.5‱): in Figure 5.6 we provide upper bounds on approximation ratios for that optimal relaxed primal value with respect to the revenue achieved by the best randomized, the best deterministic and the best full-bundling mechanisms.

5.6.1 An Explicit Construction of an Optimal Dual

A characteristic of our approach here that differentiates it from the results in the previous sections is that it is completely constructive: *we give explicit, simple closed-form definitions of the dual functions, rather than just proving their existence.* This immediately allows for a trivial check of optimality through weak duality (Lemma 3.1): just compute their (dual objective) value and see if this coincides with the revenue induced by the (primal) utility function. However, we follow an even simpler way: we deploy (tight) complementarity (Lemma 3.2) and so we can verify their optimality just by looking at some simple features of their structure.

The primal and dual programs are now of the form:

[5] An example of the necessity of convexity was also given in Section 3.3.1, even for one item, but the distribution used there is not regular.

$$\text{maximize} \qquad \int_c^{c+1}\int_c^{c+1} \frac{\partial u(\mathbf{x})}{\partial x_1}x_1 + \frac{\partial u(\mathbf{x})}{\partial x_2}x_2 - u(\mathbf{x})\,d\mathbf{x}$$

over the space of absolutely continuous functions $u : [c, c+1]^2 \longrightarrow \mathbb{R}_+$ with

$$\frac{\partial u(\mathbf{x})}{\partial x_1}, \frac{\partial u(\mathbf{x})}{\partial x_2} \leq 1, \tag{5.26}$$

for a.e. $x_1, x_2 \in [c, c+1]$, and

$$\text{minimize} \int_c^{c+1}\int_c^{c+1} z_1(\mathbf{x}) + z_2(\mathbf{x})\,d\mathbf{x}$$

over the space of absolutely continuous functions $z_1, z_2 : [c, c+1]^2 \longrightarrow \mathbb{R}_+$ with

$$\frac{\partial z_1(\mathbf{x})}{\partial x_2} + \frac{\partial z_2(\mathbf{x})}{\partial x_2} \leq 3 \tag{5.27}$$

$$z_1(c, x_2), z_2(x_1, c) \leq c, \tag{5.28}$$

$$z_1(c+1, x_2), z_1(x_1, c+1) \geq c+1, \tag{5.29}$$

for a.e. $x_1, x_2 \in [c, c+1]$. We also state the form of exact complementarity we take in our case of uniform distributions by setting $\varepsilon = 0$ in Lemma 3.2:

Lemma 5.4 (Exact Complementarity for two uniform goods)**.** *If for a.e.* $\mathbf{x} \in [c, c+1]$ *the following conditions hold for a pair of primal and dual solutions* $\tilde{u}$ *and* z_1, z_2 *then they are both optimal:*

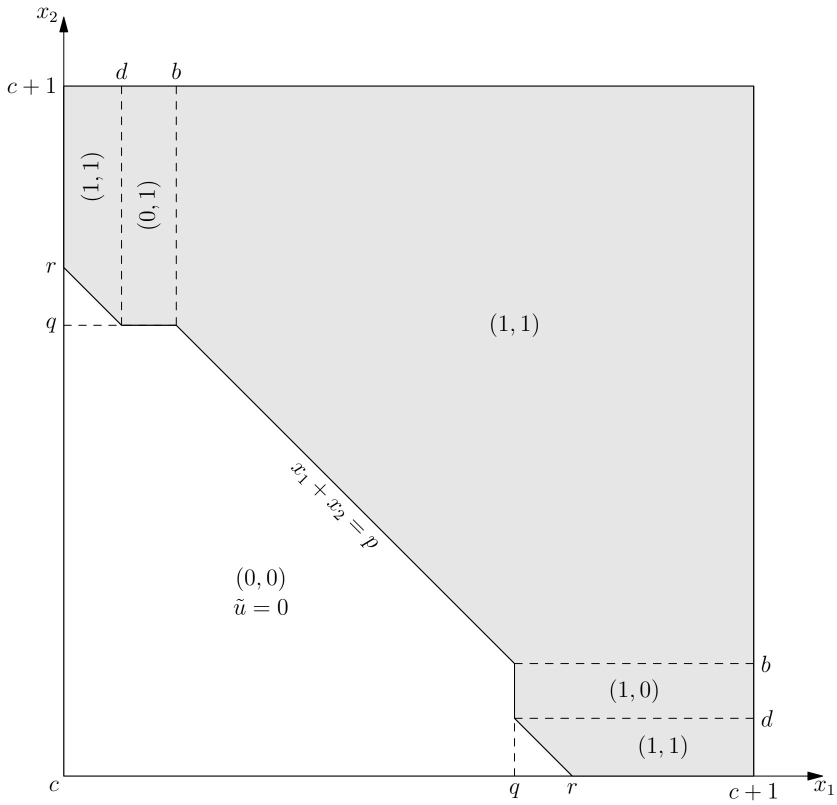
- *Either* $\tilde{u}(\mathbf{x})$ *is zero or the dual constraints* (5.27)–(5.29) *hold with strict equality*
- *For any* $j = 1, 2$*, either* $z_j(\mathbf{x})$ *is zero or the corresponding primal constraint in* (5.26) *holds with strict equality.*

5.6.2 The Case of $0 \leq c \leq 0.092$

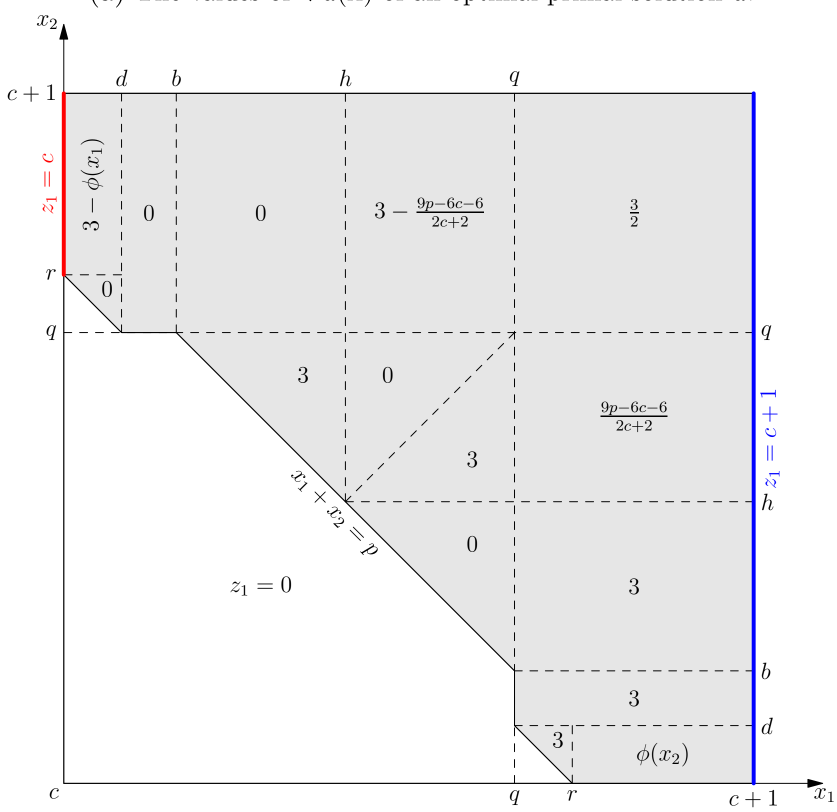
Consider the pair of primal-dual variables $\tilde{u}, (z_1, z_2)$ whose derivatives are given in Figure 5.3. The duals z_1, z_2 are symmetric, in the sense that $z_1(x_1, x_2) = z_2(x_2, x_1)$ for all $\mathbf{x}$. Notice that this is enough to completely define them, by the initial conditions $\tilde{u}(\mathbf{x}) = 0$ and $z_1(\mathbf{x}) = 0$, $z_1(\mathbf{x}) = c$ given in the white and red areas. We will argue that they are optimal. By looking at their structure in Figure 5.3, it is easy to see that the applicability of the (exact) complementarity Lemma 5.4 is just a matter of simple calculations, essentially to check constraints (5.27)–(5.29). Optimality would be immediate.

To do that, first of all we need to check that the specific parameters give rise to a consistent partitioning of the allocation space, and in particular that

$$c \leq d \leq b \leq q \leq r \leq c+1, \quad r - q = d - c \quad \text{and} \quad p = q + b.$$



(a) The values of $\nabla \tilde{u}(\mathbf{x})$ of an optimal primal solution $\tilde{u}$.



(b) The values of $\partial z_1(\mathbf{x})/\partial x_1$. The duals are symmetric and so $\partial z_2(\mathbf{x})/\partial x_2$ can be recovered by the relation $\partial z_2/\partial x_2 = 3 - \partial z_1/\partial x_1$ in the grey area (and $z_2 = 0$ in the white). Here $\phi(x) = \frac{c+1-3(x-c)}{c+1-r}$.

Figure 5.3: A pair of optimal primal-dual solutions for $0 \leq c \leq 0.092$. Notice how the primal solution $\tilde{u}$ in Figure 5.3a is *not* convex, so it does not correspond to a valid utility function of a truthful selling mechanism. The values of the various parameters are: $q = \frac{2(c+1)}{3}$, $p = \frac{4-\sqrt{2}}{3}(c+1)$, $h = \frac{p}{2}$, $b = p - q$, $r = \frac{1}{3}\left(2 + c + \sqrt{c(2+3c)}\right)$, $d = \frac{1}{3}\left(2c + \sqrt{c(2+3c)}\right)$.

Given the choice of the parameters, it is trivial to check that the two last equalities are satisfied. The first chain of inequalities is satisfied for all $0 \le c \le \bar{c}$ where

$$\bar{c} \equiv \sqrt{15 - 8\sqrt{2}} - 2\sqrt{2} + 1 \approx 0.0915.$$

At this value $c = \bar{c}$ we get the limiting situation when $d = b$ and $p = r$, and $\tilde{u}$ is a feasible utility function of the mechanism that offer only the full-bundle for a price of p. On the other hand, notice that for $c = 0$ we get $q = r$ and $d = c$, and the pair of primal-duals naturally reduces to the well-know optimal selling mechanism for two uniform items on $[0, 1]$ with prices $q = \frac{2}{3}$ and $p = \frac{4-\sqrt{2}}{3}$ for the one- and two-item bundles respectively.

We now just have to show that z_1 is feasible. In particular, it is again easy to calculate that $z_1(c + 1, x_2) = c + 1$, given the values of the parameters, the definition of ϕ and the initial condition $z_1(a, x_2) = c$ for $x_2 > r$ and $z_1(c, x_2) = 0$ otherwise. The only thing remaining to check is that z_1 never falls below zero. This can be done by easily verifying that indeed $c + \int_c^d 3 - \phi(t)\, dt = 0$, so z_1 is nonnegative at the upper critical stripe. Everywhere else, all its derivatives are nonnegative, so it cannot decrease further.

5.6.3 The Necessity of Convexity

The optimality of the solutions in Figure 5.3 is not able to directly also give us an optimal selling mechanism, because the primal solution $\tilde{u}$ constructed there is *not* convex. In fact, one can show that *no* mechanism can achieve the primal optimal objective of $\tilde{u}$, which equals

$$\frac{2}{27}\left[\left(\sqrt{2}-4\right)c^3 + 3\left(\sqrt{c(3c+2)} + \sqrt{2} + 1\right)c^2 + \left(2\sqrt{c(3c+2)} + 3\sqrt{2} + 12\right)c + \sqrt{2} + 6\right] \equiv \textsc{Opt}(c)$$

This proves that dropping the convexity constraint (in the initial formulation of our primal program) is not without loss, even in the simplest of settings: one bidder, two i.i.d. uniform items over $[c, c + 1]$ with $0 \le c \le \bar{c}$. However, it turns out that this optimal objective is still not far away from the optimal mechanism's revenue, in fact it is extremely close even to that of the best deterministic or just the best full-bundle mechanism. Specifically in Figure 5.6 one can see that the best randomized mechanism is within a factor of 7.5‱, and the best deterministic and full-bundle within factors of 2‰ and 9‰, respectively, with respect to $\textsc{Opt}(c)$.

Let's make this discussion more rigorous. As we've mentioned before, our primal-dual formulation relaxes the original optimal revenue problem in two ways: first drop the convexity constraint, that corresponds to the truthfulness requirement; then we drop the dual variables s_j that correspond to the primal constraints $\nabla u(\mathbf{x}) \ge \mathbf{0}$. The latter relaxation has no actual effect to the optimal solution of the primal-dual programs, at least for the particular case we study here of i.i.d. uniform valuations over intervals of the form $[c, c+1]$. The reason for that is simple: the optimal solution $\tilde{u}$ that we get after all relaxations satisfies $\nabla \tilde{u}(\mathbf{x}) \geq \mathbf{0}$ anyway.

So now let us focus on the necessity of the crucial remaining condition, that of convexity. By Pavlov's results we know that for $c > 0.077$ full bundling is an optimal selling mechanism. Such a mechanism in our setting sets a take-it-or-leave-it price s for both items together, thus having an expected revenue of $(1 - \frac{(s-2c)^2}{2})s$ (a simple probabilistic argument, taking into consideration the area of the grey region in that case). This is maximized for s being the root of

$$27s^3 - 108s^2c + s\left(108c^2 - 54\right) - 16c^3 + 4\sqrt{2}\sqrt{(2c^2+3)^3} + 72c = 0$$

giving a revenue of

$$\text{BRev}(c) = \frac{2}{27}\left(-4c^3 + \sqrt{2}\sqrt{(2c^2+3)^3} + 18c\right),$$

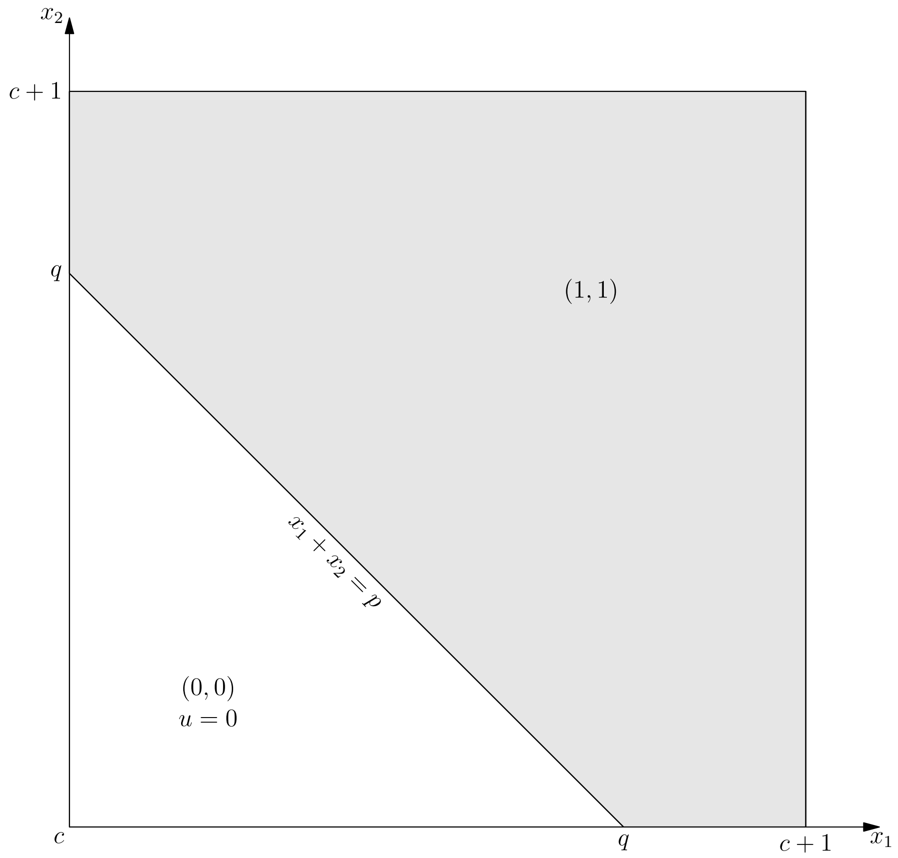
which, as we said, is also the optimal revenue for $c > 0.077$. However, it is easy to check that for all $0.077 < c < \bar{c}$ it strictly holds $\text{BRev}(c) < \text{Opt}(c)$, demonstrating the gap caused by dropping convexity.

5.6.4 The Case of $c \geq 0.092$

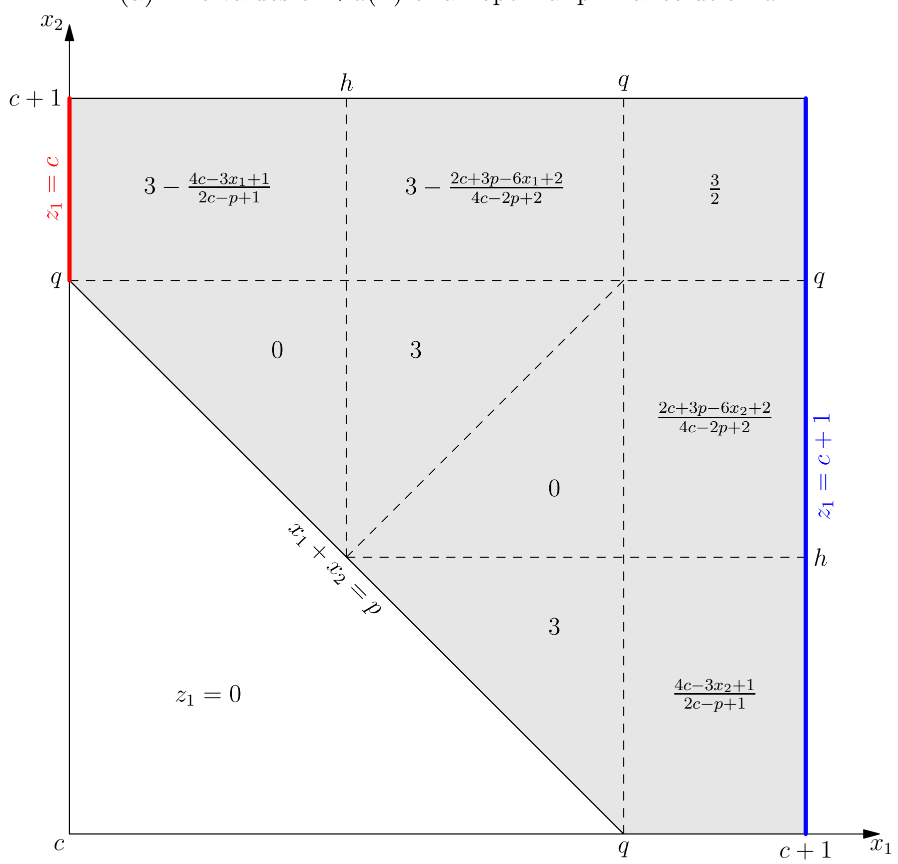
For $c \geq \bar{c}$ it turns out that convexity is indeed not needed and the optimal solution of the primal-dual programs give also the optimal selling mechanism, which is a *full-bundling* one, as can be seen by the complementarity of the pair of primal-dual solutions we propose in Figure 5.4.

It is again a matter of trivial calculations to check that indeed $z_1(c+1, x_2) = c+1$ for all $x_2 \in [c, c+1]$. The point that needs more attention is making sure that z_1 does not get negative. There is a risk of getting below 0 at the top stripe $q \leq x_2 \leq c+1$, and in particular in the box where $a \leq x_1 \leq h$. A simple derivative argument shows us that z_1 achieves a minimum there at $x_1 = x_1^* \equiv \frac{1}{3}\left(2c - 2\sqrt{4c^2+6}\right)$ and solving $z_1(x_1^*, x_2) \geq 0$ we get that $c \geq \sqrt{15 - 8\sqrt{2}} - 2\sqrt{2} + 1 = \bar{c}$, which is exactly the the case we are in, complementary to the previous Section 5.6.2.

We must mention here that there is also a more unified dual solution scheme that can cover both cases of Section 5.6.2 and Section 5.6.4 at the same time: simply replace the dual solution in Figure 5.3b for $c \leq \bar{c}$ by the slightly more involved in Figure 5.5 which however now fits smoothly with the one in Figure 5.4b for the other case of $c \geq \bar{c}$.



(a) The values of $\nabla u(\mathbf{x})$ of an optimal primal solution u.



(b) The values of $\partial z_1(\mathbf{x})/\partial x_1$. The duals are symmetric and the values of $\partial z_1(\mathbf{x})/\partial x_2$ can be recovered by the relation $\partial z_2/\partial x_2 = 3 - \partial z_1/\partial x_1$ in the grey area (and $z_2 = 0$ in the white).

Figure 5.4: A pair of optimal primal-dual solutions for $c \geq 0.092$. Notice the primal solution u in Figure 5.4a corresponds to a deterministic full-bundle mechanism. The values of the various parameters are: $p = \frac{1}{3}\left(4c + \sqrt{4c^2+6}\right)$, $q = p - c$ and $h = \frac{p}{2}$.

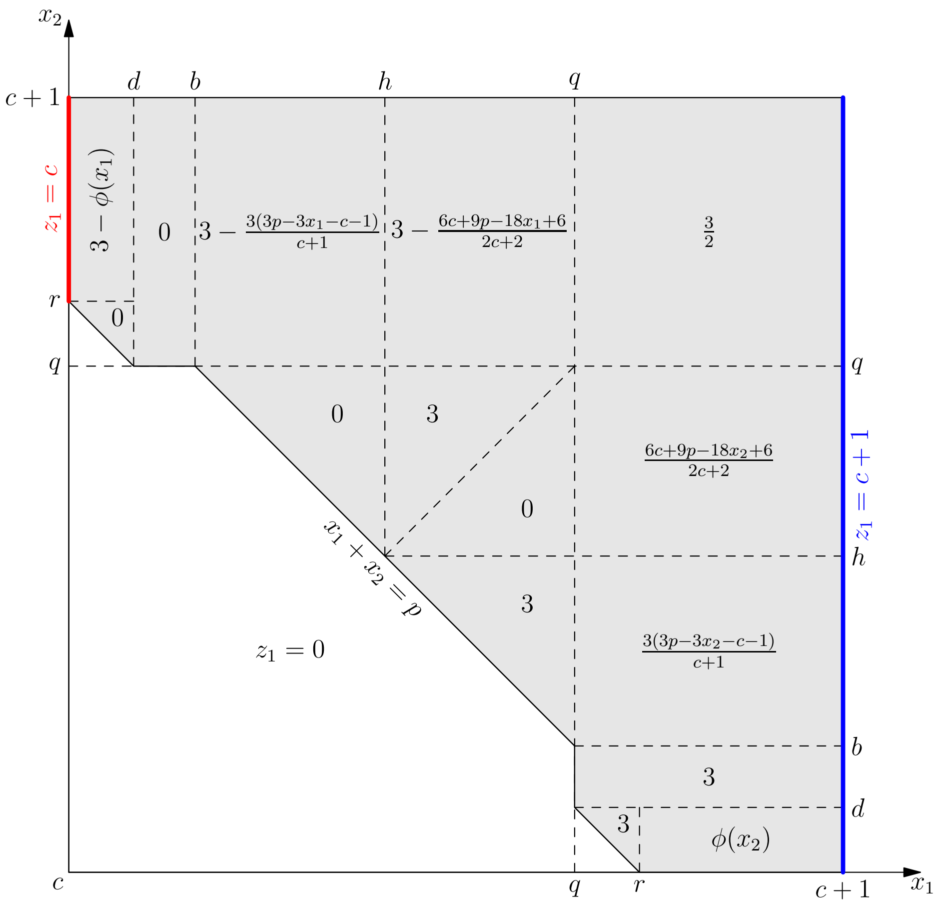
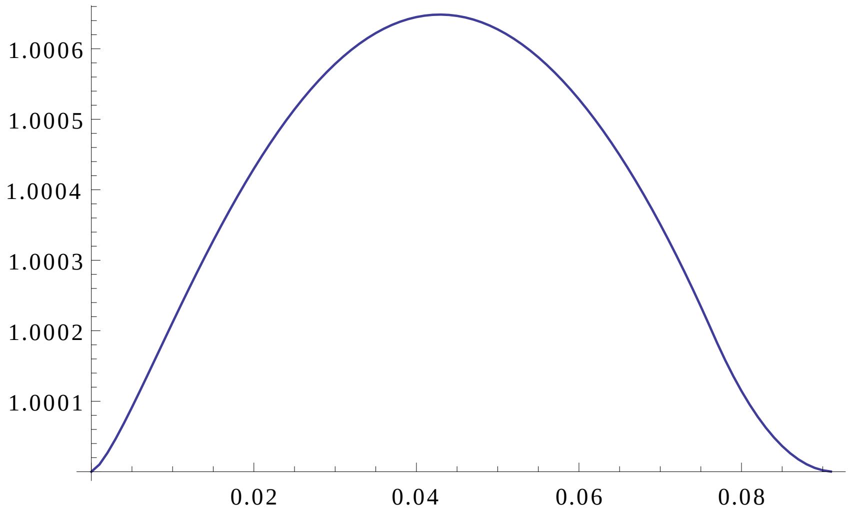
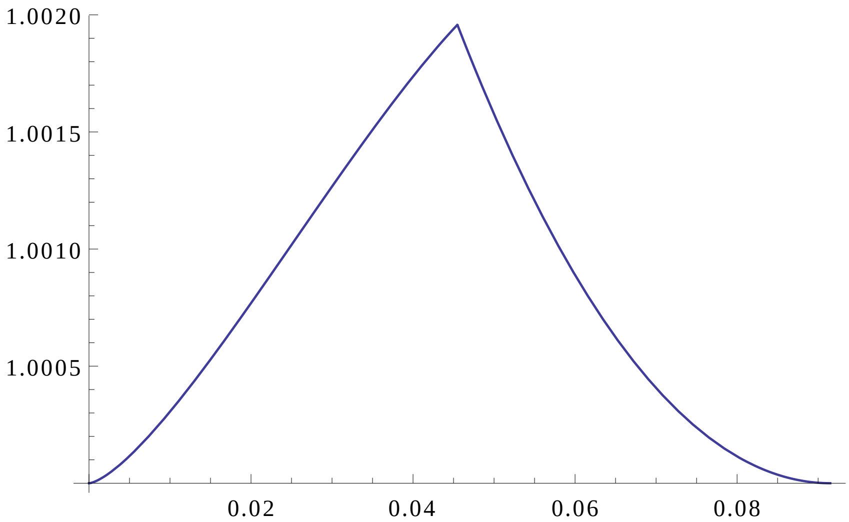


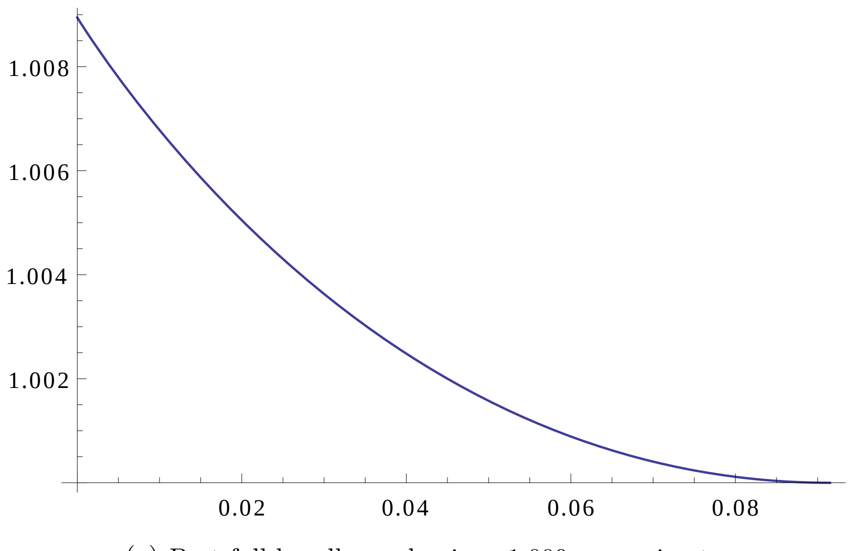
Figure 5.5: Alternative dual optimal optimal solution to the one given in Figure 5.3b. All parameters remain the same. This fits with the dual solution given in Figure 5.4b.



(a) Best randomized selling mechanism, 1.00075-approximate. To plot this one can use the form of the optimal auction given in [66].



(b) Best deterministic selling mechanism, 1.002-approximate.



(c) Best full-bundle mechanism, 1.009-approximate.

Figure 5.6: Approximation ratios of the best randomized (Figure 5.6a), deterministic (Figure 5.6b) and full-bundle (Figure 5.6c) selling mechanisms with respect to the optimal objective of the primal-dual approach with relaxed convexity, for $0 \leq c \leq \bar{c} = 0.092$.

Chapter 6

Bounding the Optimal Revenue

Following the discussion about related work in Section 2.4 as well as the development of our main results in the previous chapters, it should have been made clear by now that the problem of finding *exactly* optimal selling mechanisms for more than a single good is notoriously difficult: More than 30 year after the seminal work of Myerson [58], only the case of two and three uniformly i.i.d. goods was solved, and it is believed that closed-form descriptions are in general beyond our grasp (see e.g. [26]). Our work so far in this thesis was able to solve the case of two independent (but not necessarily identical) goods for a wide range of distributional priors, and also extend the uniform case for up to six items (with the optimality of the SJA mechanism conjectured to hold for any number of goods m, see Section 4.2).

So, it is essential to try to *approximate* the optimal revenue by selling mechanisms that are as *simple* as possible. We may lose something with respect to the total revenue objective, but on the other hand these mechanisms are much easier to understand, describe, analyze and implement, and such results may in fact enrich our understanding of the character of exact optimal auctions in general. Inspired by the elegant approach of Hart and Nisan [38] (see Section 2.4), in this chapter we take the opposite direction to their universal approximation guarantees for general independent distributions, and try to give better, specialized bounds for the case of uniform and exponential distributions. The choice of these two particular priors here is not random; we wanted to study "canonical" examples of distributions, one for bounded-interval supports and one having full support $[0, \infty)$. The uniform and exponential distributions are, respectively, the *maximum entropy* probability distributions for these two settings, intuitively being the "natural" choices if one wants to make as few assumptions as possible (see e.g. [34, Sect. 3.4.3]).

Our main strategy is driven by the standard technique in approximation algorithms, to use weak duality to upper-bound the optimal objective and then use this to calculate approximation upper bounds for particular algorithms. For this, we will once more use the duality-theory framework developed in Chapter 3. In particular, we use the weak-duality Lemma 3.1 in order to get specific closed-form bounds for our settings

(Theorems 6.1 and 6.3), by constructing and plugging-in appropriate feasible dual solutions (Theorems 6.2 and 6.4). This is the most technical part of this chapter. This technique is completely different to previous results on approximate mechanisms which rely entirely on probabilistic analysis methods (e.g. the core-tail decomposition of [48]). Our bounds on the optimal revenue are very simple expressions, depending on the number of items m. We believe that, given how difficult is the problem of *exactly* determining the optimal revenue, coming up with such formulas is a very useful tool for (approximate) auction analysis, and is of its own interest.

By comparing these bounds to the revenue obtained by the simple mechanisms studied in Hart and Nisan [38] we are able to give closed-form approximation guarantees with respect to the number of items m, in both settings that we are interested in: for the case of i.i.d. uniform distributions (see Figure 6.2) over the unit interval we show that selling the items separately is 2-approximate[1] and that selling in a full-bundle *always* performs better and is asymptotically optimal; for independent (and not necessarily identical) exponential distributions (see Figure 6.3) we give a closed-form formula upper bound (6.18) for selling separately that can be loosely[2] upper-bounded by $e \approx 2.7$.

Furthermore, if the exponential distributions are in addition identical, then we can show (Theorem 6.6) that selling deterministically in a full-bundle is optimal, for *any* number of goods. We derive this optimality as a side result of the analysis of a very simple and natural randomized selling mechanism that we propose for the setting of independent (but not necessarily identical) exponential valuations. We call it PROPORTIONAL (Definition 6.1) and allocates the items somehow proportionally with respect to every item's exponential distribution parameter. We compute the expected revenue of this mechanism (Theorem 6.5) and using again the optimal revenue bounds derived earlier, we show that PROPORTIONAL's approximation ratio is at most equal to the ratio between the maximum and minimum parameters of the independent exponential distributions. For i.i.d. settings, this ratio is of course equal to 1, proving optimality and PROPORTIONAL reduces to full-bundling.

6.1 Uniform Distributions

In this section we consider m items with i.i.d. types x_j coming from a uniform distribution over the unit interval I, i.e. $f_j(x_j) = 1$ for all $j \in [m]$. Then, the primal and dual programs are given by (4.1) and (4.2) respectively:

Theorem 6.1 (Weak Duality for uniform distributions)**.** *The dual constraints for the*

[1] This ratio of 2 is in fact *not* tight, but only asymptotically as $m \to \infty$, meaning that selling separately is even better for a small number of items.

[2] See the analogous discussion in Footnote 1.

m-items uniform i.i.d. setting over I^m become:

$$z_j(0,\mathbf{x}_{-j}) = 0, \qquad \textit{for all } j \in [m], \tag{6.1}$$

$$z_j(1,\mathbf{x}_{-j}) \geq 1, \qquad \textit{for all } j \in [m], \tag{6.2}$$

$$\sum_{j=1}^{m} \frac{\partial z_j(\mathbf{x})}{\partial x_j} \leq m+1, \tag{6.3}$$

and the dual objective upper-bounds optimal revenue:

$$\textsc{Rev}\left(\mathcal{U}^m\right) \leq \sum_{j=1}^{m} \int_{I^m} z_j(\mathbf{x})\, d\mathbf{x}. \tag{6.4}$$

Theorem 6.2. *The optimal revenue from selling m goods having uniform i.i.d. valuations over the unit interval is at most*

$$\frac{m(1+m^2)}{2(1+m)^2}.$$

Proof. Let

$$\mathcal{I}_m \equiv \{\mathbf{v} = (v_1, v_2, \dots, v_m) \mid v_j \in \{0,1\}, j \in [m]\}$$

be the set of nodes of the m-dimensional unit hypercube and for every node $\mathbf{v} \in \mathcal{I}_m$ define $L_{\mathbf{v}}$ to be the following subspace of I^m:

$$L_{\mathbf{v}} = \left\{ \mathbf{x} \in I^m \;\middle|\; x_j \in \left[0, \frac{1}{m+1}\right] \text{ if } v_j = 0 \text{ and } x_j \in \left(\frac{1}{m+1}, 1\right] \text{ if } v_j = 1, \quad j \in [m] \right\}$$

A simple observation is that $L_{\mathbf{v}}$'s form a valid partition of I^m, i.e.

$$\mathbf{v}, \mathbf{v}' \in \mathcal{I}_m \wedge \mathbf{v} \neq \mathbf{v}' \Longrightarrow L_{\mathbf{v}} \cap L_{\mathbf{v}'} = \emptyset \qquad \text{and} \qquad \bigcup_{\mathbf{v} \in \mathcal{I}_m} L_{\mathbf{v}} = I^m$$

Now we are going to construct a feasible dual solution, that is, valid z_j's, to plug them into Theorem 6.1. Fix some $j \in [m]$ and a subspace $L_{\mathbf{v}} \subseteq I^m$ (by fixing a $\mathbf{v} = (v_1, v_2, \dots, v_m) \in \mathcal{I}_m$) and define $z_j : L_{\mathbf{v}} \longrightarrow \mathbb{R}_+$ as follows:

- If $v_j = 0$, set $z_j(\mathbf{x}) = 0$ for all $\mathbf{x} \in L_{\mathbf{v}}$.
- Otherwise, i.e. if $v_j = 1$, set

$$z_j(\mathbf{x}) = \max\left\{0, \frac{m+1}{k}(x_j - c_k)\right\} = \begin{cases} 0, & \text{if } \frac{1}{m+1} < x_j \leq c_k, \\ \frac{m+1}{k}(x_j - c_k), & \text{if } c_k < x_j \leq 1, \end{cases}$$

  for all $\mathbf{x} \in L_{\mathbf{v}}$, where

$$k = k(\mathbf{v}) = \sum_{j=1}^{m} v_j \quad \text{and} \quad c_k = 1 - \frac{k}{m+1}.$$

By this construction, and by letting $\mathbf{v}$ range over $\mathcal{I}_m$, we have a well defined function $z_j : I^m \longrightarrow \mathbb{R}_+$. Each $\mathbf{x} \in [0,1]^m$ belongs to a unique partition $L_{\mathbf{v}}$ (corresponding to a *unique* $\mathbf{v} = \mathbf{v}(\mathbf{x})$), thus also well defining $k = k(\mathbf{x})$. So, the above definition can be written more compactly as

$$z_j(\mathbf{x}) = \begin{cases} 0, & \text{if } \ 0 < x_j \leq c_k, \\ \frac{m+1}{k}(x_j - c_k), & \text{if } \ c_k < x_j \leq 1. \end{cases}$$

It is easy to check, directly from this definition, that

$$z_j(0, x_{-j}) = 0 \quad \text{and} \quad z_j(1, x_{-j}) = 1 \tag{6.5}$$

for all $j = 1, 2, \ldots, m$ and $x_{-j} \in I^{m-1}$ and also that

$$\frac{\partial z_j(\mathbf{x})}{\partial x_j} = \begin{cases} 0, & \text{if } \ 0 < x_j \leq c_k, \\ \frac{m+1}{k}, & \text{if } \ c_k < x_j \leq 1. \end{cases} \tag{6.6}$$

Furthermore, if we fix some $\mathbf{x} \in I^m$ (and thus also fix the corresponding, well-defined, $\mathbf{v} = \mathbf{v}(\mathbf{x}) \in \mathcal{I}_m$ and $k = \sum_{j=1}^m v_j$), we see from property (6.6) above that

$$\sum_{j=1}^m \frac{\partial z_j(\mathbf{x})}{\partial x_j} \leq \sum_{j=1}^m v_j \frac{m+1}{k} = \frac{m+1}{k} \sum_{j=1}^m v_j = m+1. \tag{6.7}$$

But now we can see that Equations (6.5) and (6.7) are exactly properties (6.1), (6.2) and (6.3).

The last remaining step of the proof is to evaluate the dual objective and show that

$$\int_{I^m} \sum_{j=1}^m z_j(\mathbf{x})\, d\mathbf{x} = \frac{m(1+m^2)}{2(1+m)^2}.$$

Indeed

$$\begin{aligned}
\int_{I^m} \sum_{j=1}^m z_j(\mathbf{x})\, d\mathbf{x} &= \sum_{\mathbf{v} \in \mathcal{I}_m} \int_{L_{\mathbf{v}}} \sum_{j=1}^m z_j(\mathbf{x})\, d\mathbf{x} \\
&= \sum_{\mathbf{v} \in \mathcal{I}_m} \int_{L_{\mathbf{v}}} \sum_{j:\mathbf{v}_j=1} z_j(\mathbf{x})\, d\mathbf{x} \\
&= \sum_{\kappa=1}^m \sum_{\mathbf{v}:k(\mathbf{v})=\kappa} \int_{L_{\mathbf{v}}} \sum_{j:\mathbf{v}_j=1} z_j(\mathbf{x})\, d\mathbf{x} \\
&= \sum_{\kappa=1}^m \binom{m}{\kappa} \underbrace{\int_0^{\frac{1}{m+1}} \cdots \int_0^{\frac{1}{m+1}}}_{m-\kappa \text{ times}} \underbrace{\int_{\frac{1}{m+1}}^1 \cdots \int_{\frac{1}{m+1}}^1}_{\kappa \text{ times}} \sum_{j:\mathbf{v}_j=1} z_j(\mathbf{x})\, d\mathbf{x} \\
&= \sum_{\kappa=1}^m \binom{m}{\kappa} \int_0^{\frac{1}{m+1}} \cdots \int_0^{\frac{1}{m+1}} \int_{\frac{1}{m+1}}^1 \cdots \int_{\frac{1}{m+1}}^1 \sum_{j:\mathbf{v}_j=1} \frac{m+1}{\kappa}(x_j - c_k)\, d\mathbf{x}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{m} \binom{m}{k} \int_0^{\frac{1}{m+1}} \cdots \int_0^{\frac{1}{m+1}} \int_{\frac{1}{m+1}}^1 \cdots \int_{\frac{1}{m+1}}^1 \sum_{j:\mathbf{v}_j=1} \frac{m+1}{k}(x_j - c_k)\, d\mathbf{x} \\
&= \sum_{k=1}^{m} \binom{m}{k} \left(\frac{1}{m+1}\right)^{m-k} k \int_{\frac{1}{m+1}}^1 \cdots \int_{\frac{1}{m+1}}^1 \int_{c_k}^1 \frac{m+1}{k}(x - c_k)\, dx \\
&= \sum_{k=1}^{m} \binom{m}{k} \left(\frac{1}{m+1}\right)^{m-k} k \left(1 - \frac{1}{m+1}\right)^{k-1} \int_{c_k}^1 \frac{m+1}{k}(x - c_k)\, dx \\
&= \sum_{k=1}^{m} \binom{m}{k} \frac{m^{k-1}}{(m+1)^{m-2}} \int_{c_k}^1 (x - c_k)\, dx \\
&= \sum_{k=1}^{m} \binom{m}{k} \frac{m^{k-1}}{(m+1)^{m-2}} \frac{(1-c_k)^2}{2} \\
&= \frac{1}{2(m+1)^m} \sum_{k=1}^{m} \binom{m}{k} k^2 m^{k-1} \\
&= \frac{m(1+m^2)}{2(1+m)^2}.
\end{aligned}
$$

□

Discussion of Theorem 6.2 We must mention here that one can trivially get an upper bound of $\frac{m}{2}$ for the optimal revenue $\mathrm{REV}(\mathcal{U}^m)$, which is close to that given by Theorem 6.2: simply observe that by the IR constraint the seller cannot expect to extract more revenue than the buyer's surplus, i.e. the sum of the item valuations $\sum_{j=1}^{m} x_j$, which for the uniform distribution has an expectation of $\frac{m}{2}$ (see also Section 2.3.3). The two bounds are equal in the limit as the number of items grows large, however the one in Theorem 6.2 still gives an improvement by a factor of $\frac{(m+1)^2}{m^2+1}$, which especially for a small number of goods, is not insignificant. Notice that, due to the Law of Large Numbers, the optimal revenue as $m \to \infty$ will anyway tend to the expected full surplus, not only for the uniform distribution, but for any kind of independently distributed items[3].

From that perspective, we believe that it is interesting to get bounds on the optimal revenue other than the trivial ones derived from using the above surplus-bound argument. To our knowledge Theorem 6.2 is the first such result in the literature. But, probably even more important than the improvement in the bound's value itself, is the underlying technique of providing *explicit* feasible dual solutions to plug into Theorem 6.1: this can give new insights in the structure of good approximations of the optimal revenue, something which is known to be particularly difficult for our problem. To demonstrate this, consider for example the case of just two goods ($m = 2$) with valuations drawn uniformly from I. The dual program (6.1)–(6.3) essentially asks to allocate a total available value of 3 among the derivatives of z_1 and z_2 over I^2 in a way that these functions start at 0 and grow up above 1 at the boundary of I. On

[3]For more on this, see the discussion in [38]. This is the reason why, as Hart and Nisan [40] state, for our problem "the difficult case is when there is more than one but not too many goods".

one hand there is the trivial way to do that, simply allocating the total weight equally: $\frac{\partial z_1(\mathbf{x})}{\partial x_1} = \frac{\partial z_2(\mathbf{x})}{\partial x_2} = \frac{3}{2}$ for all $\mathbf{x} \in I^2$; this is clearly a suboptimal solution, since the functions end up reaching a value of $\frac{3}{2}$ at the boundary's end, way above 1. On the other hand, there is the optimal way to do it, given in [29] (see Figure 6.1b). However, this construction is rather involved and very difficult to generalize, and in fact only *existential* proofs of optimality are know for more goods, and only up to 6 items. So, it seems essential to find some middle ground within these two extremes, providing a good approximation to the optimal revenue but also still being simple enough to generalize for any number of items m. This is exactly what the construction of the dual solutions in the proof of Theorem 6.2 provides. A demonstration is given in Figure 6.1 for the case of two goods.

6.2 Exponential Distributions

In this section we consider m items with types x_j coming from independent (but not necessarily identical) exponential distributions with parameters $\lambda_j > 0$, i.e. $f_j(x_j) = \lambda_j e^{-\lambda_j x_j}$ for all $j \in [m]$ and x_j range over $[0, \infty)$. Then, the primal and dual programs from (3.4) and (3.5) give:

Theorem 6.3 (Weak Duality for exponential distributions)**.** *The dual constraints for the m-items independent exponential setting (with parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$) become:*

$$z_j(0, \mathbf{x}_{-j}) = 0, \qquad \text{for all } j \in [m], \tag{6.8}$$

$$\sum_{j=1}^{m} \frac{\partial z_j(\mathbf{x})}{\partial x_j} \leq \lambda \, (m + 1 - w) \, e^{-w}, \tag{6.9}$$

where $w = \sum_{j=1}^{m} \lambda_j x_j$, $\lambda = \prod_{j=1}^{m} \lambda_j$ and the dual objective upper-bounds optimal revenue:

$$\text{Rev}\,(\mathcal{E}) \leq \sum_{j=1}^{m} \int_{\mathbb{R}_+^m} z_j(\mathbf{x}) \, d\mathbf{x}. \tag{6.10}$$

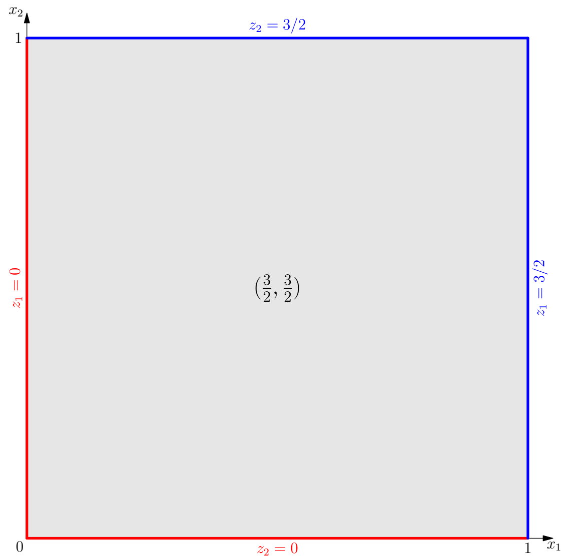
Now we will need to introduce some notation. In the following, for m positive integer and $w \in \mathbb{R}_+$ we will denote the (upper) incomplete Gamma function by

$$\Gamma(m, w) \equiv \int_w^{\infty} t^{m-1} e^{-t} \, dt = (m-1)! e^{-w} \sum_{k=0}^{m-1} \frac{w^k}{k!}$$

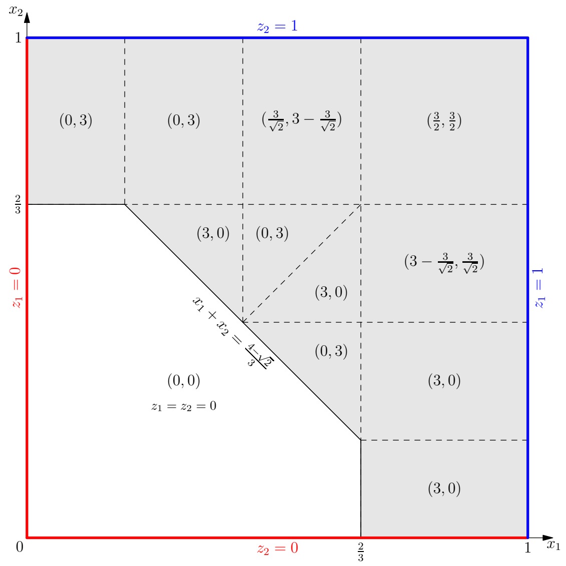
and define

$$g(m, w) = \Gamma(m+1, w) - (m+1)\Gamma(m, w). \tag{6.11}$$

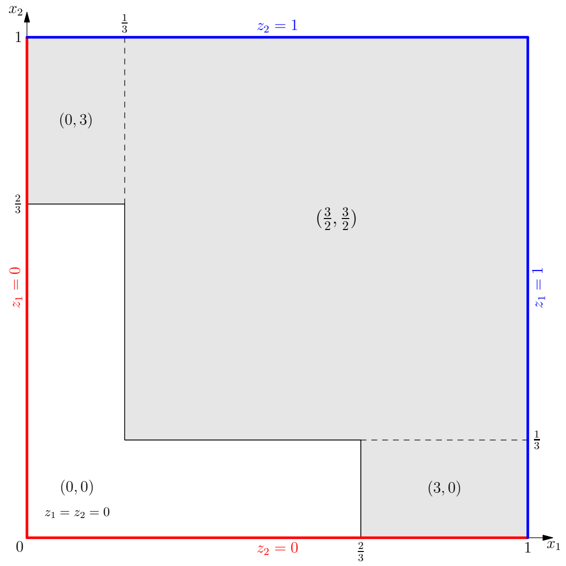
Function g is continuous and has a unique root with respect to variable $w \in \mathbb{R}_+$. Let's denote this root by γ_m^*. In fact, $\gamma_m^* \in (0, m+1)$ and also $g(m, w) > 0$ for all $w \in (\gamma_m^*, \infty)$. So, the following function on the positive integers is well defined and



(a) Trivial solution



(b) Optimal solution



(c) The solution in Theorem 6.2

Figure 6.1: Different feasible dual solutions z_1, z_2 for two goods with uniform valuations over $[0, 1]$, given by their critical derivatives $\left(\frac{\partial z_1(x_1,x_2)}{\partial x_1}, \frac{\partial z_2(x_1,x_2)}{\partial x_2}\right)$.

nonnegative:

$$G(m) \equiv \int_0^\infty \max\{0, g(m,w)\}\, dw = \int_{\gamma_m^*}^\infty g(m,w)\, dw = \gamma_m^* \Gamma(m, \gamma_m^*) = \gamma_m^{*\,m+1} e^{-\gamma_m^*}. \tag{6.12}$$

A detailed proof of all the above properties of function $g(m, w)$ and calculations can be found in the following subsection.

Properties of Function $g(m, w)$

Fix some positive integer m. Then $g(w) \equiv g(m, w)$ is an absolutely continuous function on $\mathbb{R}_+$ with derivative

$$\begin{aligned} g'(w) &= \frac{\partial \Gamma(m+1, w)}{\partial w} - (m+1)\frac{\partial \Gamma(m, w)}{\partial w} \\ &= -\frac{w^m}{e^w} - (m+1)\left(-\frac{w^{m-1}}{e^w}\right) \\ &= (m+1-w)w^{m-1}e^{-w}. \end{aligned} \tag{6.13}$$

This means that $g(w)$ is strictly increasing in $[0, m+1]$ and strictly decreasing in $[m+1, \infty)$. Also we can compute:

$$\begin{aligned} g(m, 0) &= \Gamma(m+1, 0) - (m+1)\Gamma(m, 0) \\ &= m! - (m+1)(m-1)! = -(m-1)! < 0 \\ g(m, m+1) &= \Gamma(m+1, m+1) - (m+1)\Gamma(m, m+1) \\ &= (m-1)!e^{-(m+1)}\left[m\sum_{k=0}^{m}\frac{(m+1)^k}{k!} - (m+1)\sum_{k=0}^{m-1}\frac{(m+1)^k}{k!}\right] \\ &= (m-1)!e^{-(m+1)}\sum_{k=0}^{m-1}\left(\frac{(m+1)^m}{m!} - \frac{(m+1)^k}{k!}\right) > 0 \\ \lim_{w\to\infty} g(m, w) &= 0. \end{aligned}$$

From the above we can deduce that $g(w)$ has a unique root γ^* in $\mathbb{R}_+$. In fact $\gamma^* \in (0, m+1)$ and $g(w) < 0$ for all $w \in (0, \gamma^*)$ and $g(w) > 0$ for all $w \in (\gamma^*, \infty)$.

Furthermore, we know that the incomplete gamma function has the property

$$\Gamma(m+1, w) = m\Gamma(m, w) + w^m e^{-w}. \tag{6.14}$$

With the help of this we can see that

$$\begin{aligned}\int_a^\infty \mathit{\Gamma}(m,w) &= [w\mathit{\Gamma}(m,w) - \mathit{\Gamma}(m+1,w)]_a^\infty \\ &= \Big[w\mathit{\Gamma}(m,w) - m\mathit{\Gamma}(m,w) - w^m e^{-w}\Big]_a^\infty \\ &= \Big[(w-m)\mathit{\Gamma}(m,w) - w^m e^{-w}\Big]_a^\infty \\ &= (m-a)\mathit{\Gamma}(m,a) + a^m e^{-a}\end{aligned}$$

for any positive integer m and $a \in \mathbb{R}_+$, so

$$\begin{aligned}\int_a^\infty g(m,w)\,dw &= \int_a^\infty \mathit{\Gamma}(m+1,w) - (m+1)\int_a^\infty \mathit{\Gamma}(m,w) \\ &= (m+1-a)\mathit{\Gamma}(m+1,a) + a^{m+1}e^{-a} - (m+1)(m-a)\mathit{\Gamma}(m,a) - (m+1)a^m e^{-a} \\ &= (m+1-a)(m\mathit{\Gamma}(m,a) + a^m e^{-a}) - (m+1)(m-a)\mathit{\Gamma}(m,a) + a^m e^{-a}(a-m-1) \\ &= [(m+1-a)m - (m+1)(m-a)]\,\mathit{\Gamma}(m,a) \\ &= a\mathit{\Gamma}(m,a).\end{aligned}$$

Also due to (6.14) we get

$$\begin{aligned}g(m,w) &= \mathit{\Gamma}(m+1,w) - (m+1)\mathit{\Gamma}(m,w) \\ &= m\mathit{\Gamma}(m,w) + w^m e^{-w} - (m+1)\mathit{\Gamma}(m,w) \\ &= w^m e^{-w} - \mathit{\Gamma}(m,w)\end{aligned}$$

and so, since γ^* is a root of $g(m,w)$ this means that

$$\mathit{\Gamma}(m,\gamma^*) = (\gamma^*)^m e^{-\gamma^*}.$$

We will now show the bound $\frac{G(m)}{m!} < 1$ we used in (6.18). We have:

$$\begin{aligned}\frac{G(m)}{m!} &= \frac{\gamma_m^* \mathit{\Gamma}(m,\gamma_m^*)}{m!} && \text{from (6.12)} \\ &= e^{-\gamma_m^*}\frac{\gamma_m^*}{m}\sum_{k=0}^{m-1}\frac{{\gamma_m^*}^k}{k!} \\ &< e^{-\gamma_m^*}\sum_{k=0}^{m-1}\frac{{\gamma_m^*}^{k+1}}{(k+1)!} && \text{since } \ k+1 \le m \\ &< e^{-\gamma_m^*}\sum_{k=0}^{\infty}\frac{{\gamma_m^*}^k}{k!} = 1.\end{aligned}$$

We are now ready to upper bound the optimal revenue in the case of multiple exponentially distributed goods:

Theorem 6.4. *The optimal revenue from selling m goods having independent exponential valuations (with parameters $\lambda_1, \lambda_2, \dots, \lambda_m$) is at most*

$$\frac{G(m)}{m!}\sum_{j=1}^{m}\frac{1}{\lambda_j},$$

where $G(m)$ is defined in (6.12). In the special case of i.i.d. exponential valuations with parameter λ this becomes

$$\frac{G(m)}{(m-1)!\lambda}.$$

Proof. We will construct appropriate dual variables z_j that satisfy (6.8) and (6.9) to plug into the Weak Duality Theorem 6.3. For all $j = 1, 2, \dots, m$ and $\mathbf{x} \in \mathbb{R}_+^m$ we define

$$z_j(\mathbf{x}) = \max\left\{0, \hat{\lambda} x_j w^{-m} g(m,w)\right\} = \begin{cases} \hat{\lambda} x_j w^{-m} g(m,w), & \text{if } \ w \geq \gamma_m^* \\ 0, & \text{otherwise,} \end{cases}$$

where $\hat{\lambda} = \prod_{j=1}^m \lambda_j$, $w = \sum_{j=1}^m \lambda_j x_j$ and function $g(m,w)$ as defined in (6.11). The nonnegativity of the dual variables as well as their absolute continuity is immediate from the properties of function $g(m,w)$. It is also trivial to see that condition (6.8) is immediately satisfied by the definition of z_j. Regarding condition (6.9), for any $j \in [m]$ and $\mathbf{x} \in \mathbb{R}_+^m$ such that $w \geq \gamma_m^*$ we calculate

$$\begin{aligned}
\frac{\partial z_j(\mathbf{x})}{\partial x_j} &= \hat{\lambda} w^{-m} g(m,w) + \hat{\lambda} x_j \frac{\partial w^{-m} g(m,w)}{\partial x_j} \\
&= \hat{\lambda} w^{-m} g(m,w) + \hat{\lambda}\lambda_j x_j \frac{\partial w^{-m} g(m,w)}{\partial w} \\
&= \hat{\lambda} w^{-m} g(m,w) + \hat{\lambda}\lambda_j x_j \left[w^{-m}\frac{\partial g(m,w)}{\partial w} - m w^{-m-1} g(m,w)\right] \\
&= \hat{\lambda} w^{-m} g(m,w) + \hat{\lambda}\lambda_j x_j \left[(m+1-w) w^{-1} e^{-w} - m w^{-m-1} g(m,w), \right]
\end{aligned}$$

so, by summing up we get

$$\begin{aligned}
\sum_{j=1}^{m} \frac{\partial z_j(\mathbf{x})}{\partial x_j} &= m\hat{\lambda} w^{-m} g(m,w) + \hat{\lambda} w \left[(m+1-w) w^{-1} e^{-w} - m w^{-m-1} g(m,w)\right] \\
&= \hat{\lambda}(m+1-w)e^{-w}.
\end{aligned}$$

At the remaining case of $w < \gamma_m^*$, we have that $z_j(\mathbf{x}) = 0$ for all $j \in [m]$. Also, since $\gamma_m^* < m+1$, we know that $w < m+1$, so

$$\sum_{j=1}^{m} \frac{\partial z_j(\mathbf{x})}{\partial x_j} = 0 < \hat{\lambda}(m+1-w)e^{-w}.$$

Thus, in any case (6.9) is satisfied.

Finally, we compute the dual objective value in (6.10). First notice that

$$\sum_{j=1}^{m} z_j(\mathbf{x}) = \begin{cases} \hat{\lambda} w^{-m} g(m,w) \sum_{j=1}^{m} x_j, & \text{if } \ w \geq \gamma_m^* \\ 0, & \text{otherwise.} \end{cases}$$

We perform the following change of variables in the integral:

$$x_1 = t_1 \frac{w}{\lambda_1}, x_2 = t_2 \frac{w}{\lambda_2}, \dots, x_{m-1} = t_{m-1} \frac{w}{\lambda_{m-1}} \quad \text{and} \quad x_m = (1 - t_1 - t_2 - \dots - t_{m-1}) \frac{w}{\lambda_m} \tag{6.15}$$

where $w = \sum_{j=1}^{m} \lambda_j x_j \in \mathbb{R}_+$ and $t_1, \dots, t_{m-1} \in \mathbb{R}_+$ with $0 \leq t_1 + \dots + t_{m-1} \leq 1$. Denote this subspace of I^{m-1} where t_j's range by $\mathcal{A}$. The Jacobian of this transformation equals $\frac{w^{m-1}}{\hat{\lambda}}$ and so the integral in (6.10) can be written as:

$$\begin{aligned}
\int_{\mathbb{R}_+^m} \sum_{j=1}^{m} z_j(\mathbf{x})\, d\mathbf{x} &= \int_{\gamma_m^*}^{\infty} \int_{\mathcal{A}} \lambda w^{-m} g(m,w) \sum_{j=1}^{m} x_j \cdot \frac{w^{m-1}}{\hat{\lambda}}\, dt_1\, dt_2 \dots\, dt_{m-1}\, dw \\
&= \int_{\gamma_m^*}^{\infty} g(m,w)\, dw \int_{\mathcal{A}} \frac{t_1}{\lambda_1} + \dots + \frac{t_{m-1}}{\lambda_{m-1}} + \frac{1 - t_1 - \dots - t_{m-1}}{\lambda_m}\, dt_1 \dots\, dt_{m-1} \\
&= G(m) \int_{\mathcal{A}} \sum_{j=1}^{m-1} \left(\frac{1}{\lambda_j} - \frac{1}{\lambda_m} \right) t_j + \frac{1}{\lambda_m}\, dt_1 \dots\, dt_{m-1} \\
&= G(m) \left[\frac{1}{m!} \sum_{j=1}^{m-1} \left(\frac{1}{\lambda_j} - \frac{1}{\lambda_m} \right) + \frac{1}{(m-1)! \lambda_m} \right] \\
&= \frac{G(m)}{m!} \sum_{j=1}^{m} \frac{1}{\lambda_j}.
\end{aligned}$$

At the third equation above we used the following Lemma describing some known "geometric" properties of the body $\mathcal{A}$ used in the transformation (6.15).

Lemma 6.1. *For any positive integer m,*

$$\int_{\mathcal{A}} 1\, dt_1 \dots\, dt_{m-1} = \mu(\mathcal{A}) = \frac{1}{(m-1)!} \quad \textit{and} \quad \int_{\mathcal{A}} t_j\, dt_1 \dots\, dt_{m-1} = \frac{1}{m!},$$

for all $j \in [m-1]$ (where μ denotes the standard Lebesgue measure).

□

6.3 Simple Auctions for Many Items

6.3.1 Simple, Closed-form Approximation Guarantees

Regarding i.i.d. uniform goods, Theorem 6.2 combined with (2.12) immediately gives us the following approximation ratio bound for the simple deterministic mechanism

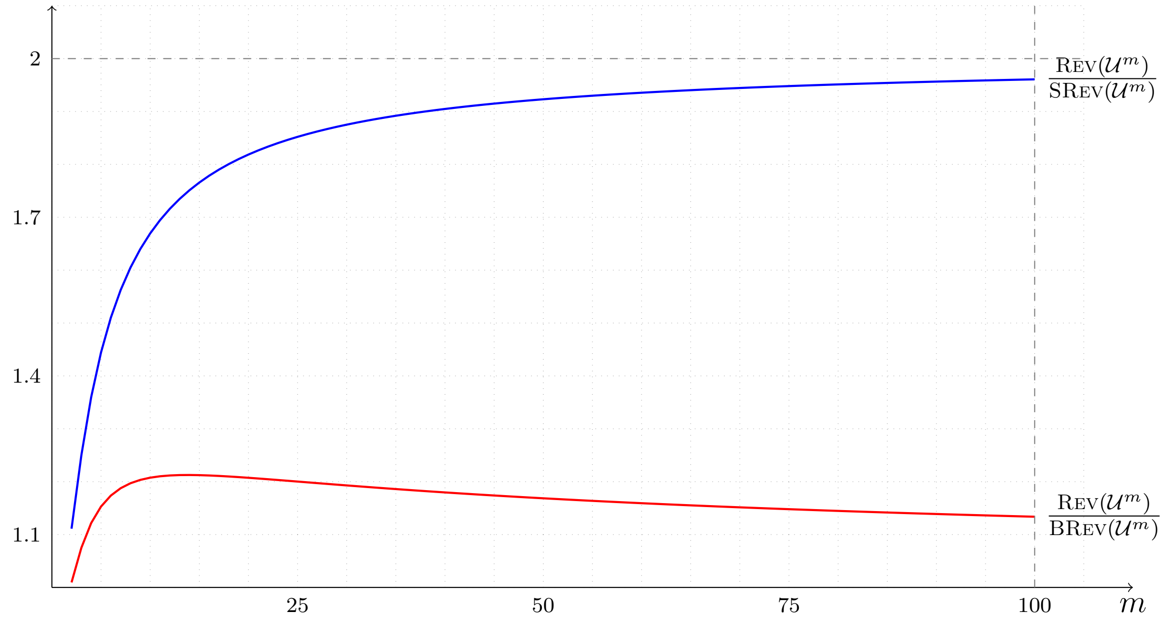


Figure 6.2: The approximation ratio bounds for the uniform i.i.d. separate and full-bundle selling mechanisms in (6.16) and (6.17).

that sells the items separately:

$$\frac{\text{Rev}(\mathcal{U}^m)}{\text{SRev}(\mathcal{U}^m)} \leq 2\frac{1+m^2}{(1+m)^2} < 2. \tag{6.16}$$

A plot of this approximation ratio for the values of $m = 1, 2, \ldots, 100$ can be found in Figure 6.2, drawn with blue colour. In the same way, using the other expression of (2.12) together with (2.13) we get a bound for the approximation ratio of the deterministic full bundle mechanism, which is asymptotically optimal:

$$\frac{\text{Rev}(\mathcal{U}^m)}{\text{BRev}(\mathcal{U}^m)} \leq \frac{m(1+m^2)}{2(1+m)^2 \sup_{x\in[0,m]} x\left(1 - \frac{1}{m!}\sum_{k=0}^{\lfloor x\rfloor}(-1)^k\binom{m}{k}(x-k)^m\right)} \to 1, \tag{6.17}$$

as $m \to \infty$. A plot of this approximation ratio for the values of $m = 1, 2, \ldots, 100$ can be found in Figure 6.2, drawn with red colour. *Notice how full bundling outperforms selling separately for* any *number of goods m.* For exponentially distributed goods, an immediate result of Theorem 6.4 combined with formula (2.14) is that for independent (not necessarily identical) exponential valuations the approximation ratio of the deterministic mechanism that sells items separately is at most

$$\frac{\text{Rev}\,(\mathcal{E})}{\text{SRev}(\mathcal{E})} \leq \frac{G(m)}{m!}e < e. \tag{6.18}$$

A plot of this approximation ratio $\frac{G(m)}{m!}e$ for the values $m = 2, 3, \ldots, 100$ can be found in Figure 6.3. The loose constant factor bound $\frac{G(m)}{m!} < 1$ is straightforward and the proof can be found in Section 6.2.

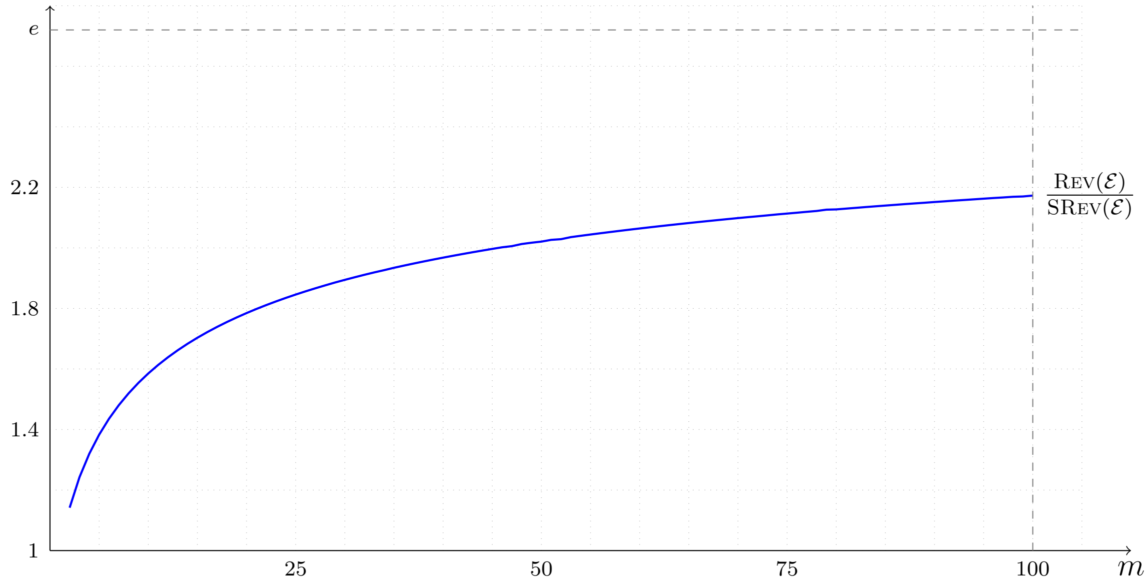


Figure 6.3: The approximation ratio bound in (6.18) for the separate selling mechanism for independent exponential valuations.

6.3.2 A Simple Randomized Selling Mechanism

Consider the following very simple randomized mechanism for the setting of independent exponential valuations with parameters $\lambda_1, \dots, \lambda_m$. Without loss of generality, in the following let us assume that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$. We will again be using our notation of $w = \sum_{j=1}^{m} \lambda_j x_j$ and $\hat{\lambda} = \prod_{j=1}^{m} \lambda_j$.

Definition 6.1 (Mechanism Proportional)**.** Sell item j with probability $\frac{\lambda_j}{\lambda_1}$ and collect a total payment of γ_m^*/λ_1 (parameter γ_m^* is defined before (6.12)).

Essentially we sell the items with probability proportional to their parameters, normalized by the largest parameter λ_1. This mechanism is truthful, because it corresponds to the following utility function

$$u(\mathbf{x}) = \max\left\{0, x_1 + \frac{\lambda_2}{\lambda_1}x_2 + \dots + \frac{\lambda_m}{\lambda_1}x_m - \frac{\gamma_m^*}{\lambda_1}\right\}$$

which is obviously convex and we will use the shorthand notation

$$U(w) = u(\mathbf{x}) = \max\left\{0, \frac{w}{\lambda_1} - \frac{\gamma_m^*}{\lambda_1}\right\} \tag{6.19}$$

when this is more comfortable. Now let us compute Proportional's expected revenue. By (3.1) and the fact that $f_j(x_j) = \lambda_j e^{-\lambda_j x_j}$ this is

$$\hat{\lambda} \int_{\mathbb{R}_+^m} \left(\sum_{j=1}^{m} \frac{\partial u(\mathbf{x})}{\partial x_j} - u(\mathbf{x}) \right) e^{-\sum_{j=1}^{m} \lambda_j x_j} \, d\mathbf{x}$$

and by a simple integration by parts (see e.g. the derivation in [25, Section 2]) this can be written as

$$\hat{\lambda} \int_{\mathbb{R}_+^m} u(\mathbf{x}) \left(\sum_{j=1}^{m} \lambda_j x_j - (m+1) \right) e^{-\sum_{j=1}^{m} \lambda_j x_j} \, d\mathbf{x} = \hat{\lambda} \int_{\mathbb{R}_+^m} u(\mathbf{x}) \left(w - (m+1) \right) e^{-w} \, d\mathbf{x}$$

and by performing the same change of variables as in (6.15) in the proof of Theorem 6.4 we get that PROPORTIONAL's expected revenue is

$$\hat{\lambda} \int_0^{\infty} \int_{\mathcal{A}} U(w) \left(w - (m+1) \right) e^{-w} \frac{w^{m-1}}{\lambda} \, dw \, dt_1 \, \dots \, dt_{m-1}$$

which equals

$$\frac{1}{(m-1)!} \int_0^{\infty} U(w) w^{m-1} \left(w - (m+1) \right) e^{-w} \, dw$$

by using Lemma 6.1. Utilizing (6.13) and taking into consideration that $U(0) = 0$ and $\lim_{w \to \infty} g(m, w) U(w) = 0$, integrating by parts the revenue becomes

$$\frac{1}{(m-1)!} \int_0^{\infty} U'(w) g(m, w) \, dw = \frac{1}{(m-1)!} \int_{\gamma_m^*}^{\infty} \frac{1}{\lambda_1} g(m, w) \, dw = \frac{G(m)}{(m-1)! \lambda_1},$$

for the first equation using (6.19). So we showed the following:

Theorem 6.5. *For m goods with independent (but not necessarily identical) exponential valuations (with parameters $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$), mechanism* PROPORTIONAL *has an expected revenue of*

$$\frac{G(m)}{(m-1)! \lambda_1},$$

where $G(m)$ is defined in (6.12).

Immediately, by combining Theorem 6.5 with Theorem 6.4, we get that PROPORTIONAL's approximation ratio is upper-bounded by

$$\frac{1}{m} \left(1 + \frac{\lambda_1}{\lambda_2} + \cdots + \frac{\lambda_1}{\lambda_m} \right) \leq \max_j \frac{\lambda_1}{\lambda_j} = \frac{\lambda_1}{\lambda_m}. \tag{6.20}$$

The performance of this approximation ratio bound depends heavily on the choice of the parameters λ_j. Obviously, the closer these parameters are the better the bound. However, if $\lambda_1 \gg \lambda_m$ then this ratio can be unbounded. In such a case though, we can fall back to using the constant approximation separate selling mechanism in (6.18) which is at most e-approximate.

6.3.3 An Exact Optimality Result

A very interesting consequence of (6.20) is for the special case of i.i.d. exponential priors, i.e. when $\lambda_1 = \cdots = \lambda_m = \lambda$. In that case, by (6.19) it is straightforward to see that PROPORTIONAL reduces to the simple deterministic mechanism that sells all items in a full bundle for a price of γ_m^*/λ and also the approximation ratio in (6.20) becomes 1, meaning that full bundling is optimal:

Theorem 6.6. *Selling deterministically in a full bundle*[4] *is optimal for* any number *of exponentially i.i.d. goods.*

[4]The optimal bundle price is γ_m^*/λ, where λ is the parameter of the exponential distribution and γ_m^* is given before (6.12).

Chapter 7

Future Directions

Here we present what we consider to be some interesting directions for possible future research, related to the results presented in this thesis. Although the duality-theory framework we presented here was a first step towards breaking the barrier between Myerson's single-good model and multidimensional settings, we are still far from a satisfactory general understanding of optimal auctions when multiple items are involved. In the last few years there has been a successful revival of interest in the subject, both from the computer science and economics communities, producing some very interesting results (see Section 2.4). We are hopeful that, if this effort continues with the same intensity, soon enough we might witness the development of a solid, unifying theory for multidimensional revenue maximization.

1. *(Multiple-bidder settings)* All *exact optimality* results that we are aware of, including the ones presented in this thesis, that involve multiple items, are for settings of a *single buyer*. Of course, there is the very interesting recent result of Yao [80] but this provides us only with constant *approximations* of the optimal revenue. Furthermore, the techniques used there are probabilistic in nature, most notably the core-tail decomposition introduced in [48], and it seems that in order to derive exact (or near optimal) results a different, explicit and *structural* understanding of the key characteristics of revenue maximizing auctions is needed. For example, *what is the optimal auction for selling two goods with uniform i.i.d. values to two buyers?* Currently, we don't even know how to solve this (seemingly) very simple problem. *What would be a good dual solution in this case?*

2. *(Limits of determinism)* As we mentioned in Section 2.4 and also saw in the results of Chapter 5, lotteries are in general necessary in order to achieve optimality. But what is the actual gap between randomization and determinism, at least for most "natural" settings? As Hart and Nisan [39] show, that gap can be arbitrarily unbounded in general; however, this is demonstrated for *correlated* items and it is essentially a consequence of the limited menu-size that deterministic mechanisms can offer, rather than the lack of random choices itself (see Hart

and Nisan [40] for a discussion of this). Even the logarithmic impossibility result of Babaioff et al. [6] for independent items is for many bidders, for the more restricted class of Partition deterministic auctions, and for a somehow *exotic* distribution that uses point masses and assigns almost all probability on point 0. So, we can start by asking for a *lower bound on the performance of deterministic selling mechanisms against a single buyer and i.i.d. values coming from continuous, regular distributions over real intervals*. Or, what if we restrict this question within simpler classes of deterministic mechanisms, like the ones studied in [6] or the extremely simple (and economists' favourite) full-bundling? (See also the discussion in point 7 below.)

3. *(Power of determinism)* Following point 2 above, the fact that there are some separation results, mostly in special, highly item-correlated settings, does not mean that our hope for well-performing simple deterministic selling mechanisms is lost. In the contrary actually, the recent constant approximation results of Babaioff et al. [6] and Yao [80] point in the opposite direction. However, they are not yet satisfactory enough from a practical point of view: the fact that full bundling or selling separately can perform at most 6 times worse than the optimal selling mechanism, might be of limited interest to a real-life auction designer. However, *a result in the spectrum of a 90% guaranteed performance would be extremely interesting*. The careful reader would have already glimpsed indications of such efficiency of simple deterministic auctions in some parts of this thesis (see e.g. Figure 6.2 and Section 5.5.1).

4. *(Non-regular distributions)* Given independence of the valuation priors, the regularity of the distributions (see Definition 2.6) itself is not a real issue for the classical results of Myerson [58]; there is an *ironing* process that can simulate smoothness of the critical virtual value functions. All known results for optimality in multidimensional auctions, however, do not rely on some unified generalization of these features, but rather in ad hoc handling of distributions that satisfy some form of regularity a priori. So, a very important challenge would be to *generalize in the "right" way the idea of Myerson's virtual valuations and ironing process in order to handle multiple items.*

5. *(Correlation)* The duality-theory framework that we proposed can readily, in principle, be used to handle arbitrary correlation among the value priors for the goods (see Section 3.1). Using that formulation, can we use the experience from the item-independent solutions of the examples presented throughout the chapters of this thesis, in order to construct good dual solutions to models involving even limited correlation, e.g. *common base-value distributions* [16] ? What about constant approximations by simple deterministic auctions, generalizing the results of [6]?

6. *(Computational complexity)* Although in this work we didn't directly deal with computational complexity issues, this is an important consideration to make that, in addition to the obvious intractability concerns, can also have strong conceptual correlations in the design of auctions (see also the discussion in point 7 below). By the work of Chen et al. [17] and Daskalakis et al. [26] we already know that both finding the optimal deterministic pricing rule for unit-demand valuations or the optimal randomized selling mechanism for additive ones, even in settings with a single buyer and independently distributed item values, are computationally hard. But *what is the complexity of finding the optimal* deterministic *selling mechanism for additive valuations? What about the complexity when restricted to a specific class of* simple *deterministic mechanisms? Or to randomized auctions with limited menu-size complexity [39]?* These are important questions, the answers to which would be of particular interest to economists.

7. *(Conceptual complexity)* Hardness results about optimal auctions with respect to traditional computational complexity notions [17, 24, 26] are obviously an important indication of the general difficulty of the problem of multidimensional revenue maximization. But this is only one side of the story. For instance, the discovery of a PTAS for an auction setting, does not necessarily mean tractability in terms of understanding the structure, the characteristics, or even the description, of an actual optimal auction. This is a feature that Hart and Nisan [40] refer to as *conceptual complexity*. It is to a great extent intuitive, and has to yet be formalized through the results that are to come; for instance a first attempt towards that is the menu-size complexity of Hart and Nisan [39]. But consider for example a setting of m goods and a single buyer. One of the most natural and straightforward selling strategies to investigate is placing prices on bundles of items depending only on their cardinalities [19]. The menu-size of this auction is *exponential*, namely 2^m, however conceptually is a very simple one: we only need to set m different prices. *What is the computational complexity of such* conceptually simple *mechanisms? Can we provide optimal solutions for them?* The investigation of *simple* mechanisms is essential for many reasons, for instance their ease of description and implementation, their transparency and the potential for straightforward analysis.

8. *(Beyond additive valuations)* Throughout this thesis we have assumed that the players have additive valuations. However, as was briefly mentioned in Section 2.3, there are other types of valuation functions that are of interest, most notably unit-demand ones. These have been studied extensively as well, and there are some recent exact optimality results (see e.g. the PhD thesis of Haghpanah [36]). However, there is still essentially no *unified* approach between the two classes of preferences. Thus, a natural research direction would be that of

developing a duality-theory framework, similar in character to the one presented here, for the case of unit-demand valuations. Also, what about classes of valuations that generalize *both* additive and unit-demand ones, like k-demand or gross-substitutes? This is an open question posed by Babaioff et al. [6] in the context of constant approximations from simple mechanisms, but we would like to put it forward with respect to *exact* optimality as well.

9. *(Other objectives).* The techniques developed here, and in particular the emphasis that the duality formulation puts on the *analytic* aspects of utility functions, might turn out to be useful in tackling other important mechanism design objectives, e.g. minimizing the makespan or the max-min fairness in the problem of *scheduling unrelated parallel machines* [18]; hints towards such a unified, high-level approach can be found for example in the work of Cai et al. [14].

Appendix

A.1 Exact Computation of the Prices for up to 6 Dimensions.

To decongest notation, we will drop the subscript (m) because it will be clear in which dimension we are working in and we denote $v(\alpha_1, \ldots, \alpha_r) = |V(\alpha_1, \ldots, \alpha_r)|$ where this body is defined in (4.4) and, as we mentioned after Definition 4.1, we can use the equivalent to (4.3) condition

$$v(p_1, \ldots, p_r) = rk$$

to determine the SJA payments. Also, we set $k = \frac{1}{m+1}$ throughout this section.

- $r = 1$ and any m: As we said before, it is easy to see that for any dimension m,

$$v(p_1) = 1 - p_1. \tag{A.1}$$

  From this, and applying the transformation (4.6), we solve

$$v(p_1) = 1 \cdot k \iff 1 - (1 - \mu_1 k) = k \iff \mu_1 = 1.$$

  So

$$p_1^{(m)} = \frac{m}{m+1} \quad \text{and} \quad \mu_1 = 1.$$

For higher orders $r > 1$ we can utilize the recursive way of computing the expressions for the volumes v_r, given by formula (4.5) and the initial condition (A.1).

- $r = 2$ and any m: Using the recursive formula (4.5) and (A.1) we can compute that for every p_2 such that $0 \leq p_2 - p_1 \leq p_1$ it would be

$$v(p_1, p_2) = \int_0^{p_2 - p_1} v(p_1)\, dt + \int_{p_2 - p_1}^{p_1} v(p_2 - t)\, dt + \int_{p_1}^{1} dt = p_1^2 + \frac{p_2^2}{2} - 2p_1 p_2 + 1$$

  and by applying the transformation (4.6) and plugging in the already computed value $\mu_1 = 1$ from the previous order $r = 1$, we get

$$v(1 - \mu_1 k, 2 - \mu_2 k) = 2k \iff \mu_2^2 - 4\mu_2 + 2 = 0 \tag{A.2}$$

If we pick the largest root of this equation $\mu_2 = 2+\sqrt{2}$ we can see that indeed condition $0 \le p_2 - p_1 \le p_1$ is respected (it is equivalent to $0 \le k \le 1/(1+\sqrt{2})$ which holds since $k \le \frac{1}{r+1}$ and $r \ge 2$), so we have computed that for any m

$$p_2^{(m)} = \frac{2m-\sqrt{2}}{m+1} \quad \text{and} \quad \mu_2 = 2+\sqrt{2} \approx 3.41421.$$

- $r=3$ and any m: In the same way, using again recursive formula (4.5) and the volume of the previous order $r=2$ from (A.2) we can compute that for every p_3 such that $0 \le p_3 - p_2 \le p_2 - p_1$ it would be

$$\begin{aligned} v(p_1,p_2,p_3) &= \int_0^{p_3-p_2} v(p_1,p_2)\,dt + \int_{p_3-p_2}^{p_2-p_1} v(p_1,p_3-t)\,dt + \int_{p_2-p_1}^{p_1} v(p_2-t,p_3-t)\,dt + \int_{p_1}^{1} dt \\ &= \frac{1}{6}\left(-3p_1^3 + 9p_1^2p_3 + 9p_1\left(2p_2^2 - 4p_2p_3 + p_3^2\right) - 6p_2^3 + 9p_2^2p_3 - p_3^3 + 6\right) \quad \text{(A.3)} \end{aligned}$$

and by applying the transformation (4.6) and plugging in the already computed values for $\mu_1 = 1$ and $\mu_2 = 2+\sqrt{2}$ from the previous orders, we get

$$v(1-\mu_1 k, 2-\mu_2 k, 3-\mu_3 k) = 3k \iff \mu_3^3 - 9\mu_3^2 + 9\mu_3 + 12\sqrt{2} + 15 = 0 \quad \text{(A.4)}$$

If we pick the largest again root of this equation

$$\mu_3 = \sqrt[3]{6-6\sqrt{2}+6i\sqrt{3+2\sqrt{2}}} + \frac{6^{2/3}}{\sqrt[3]{1-\sqrt{2}+i\sqrt{3+2\sqrt{2}}}} + 3 \approx 7.09717$$

we can see that indeed condition $0 \le p_3 - p_2 \le p_2 - p_1$ is respected (it is equivalent to $0 \le k \le 0.271521$ which holds since $k \le \frac{1}{r+1}$ and $r \ge 3$), so we have computed that for any m

$$p_3^{(m)} \approx 3 - \frac{7.09717}{m+1} \quad \text{and} \quad \mu_3 \approx 7.09717.$$

- $r=4$ and any m: Continuing up the same way, we compute that for every p_4 such that $0 \le p_4 - p_3 \le p_3 - p_2$ it is

$$\begin{aligned} v(p_1,p_2,p_3,p_4) &= \int_0^{p_4-p_3} v(p_1,p_2,p_3)\,dt + \int_{p_4-p_3}^{p_3-p_2} v(p_1,p_2,p_4-t)\,dt \\ &+ \int_{p_3-p_2}^{p_2-p_1} v(p_1,p_3-t,p_4-t)\,dt + \int_{p_2-p_1}^{p_1} v(p_2-t,p_3-t,p_4-t)\,dt + \int_{p_1}^{1} dt \end{aligned}$$

which equals

$$\frac{1}{24}\left(4p_1^4-16p_1^3p_4-24p_1^2\left(3p_3^2-6p_3p_4+2p_4^2\right)-16p_1\left(3p_2^3-9p_2^2p_4-9p_2\left(2p_3^2-4p_3p_4+p_4^2\right)\right.\right.$$
$$\left.\left.+6p_3^3-9p_3^2p_4+p_4^3\right)+18p_2^4-48p_2^3p_4-36p_2^2\left(2p_3^2-4p_3p_4+p_4^2\right)+12p_3^4-16p_3^3p_4+p_4^4+24\right). \tag{A.5}$$

By applying the transformation (4.6), plugging in the already computed values for $\mu_1 = 1$ and $\mu_2 = 2+\sqrt{2}$ and using the fact that μ_3 is the root of Equation (A.4), we get that equation $v(1-\mu_1 k, 2-\mu_2 k, 3-\mu_3 k, 4-\mu_4 k) = 4k$ is equivalent to

$$\mu_4^4-16\mu_4^3+24\mu_4^2+96\sqrt{2}\mu_4+128\mu_4+72\mu_3^2-144\sqrt{2}\mu_3-288\mu_3+48\sqrt{2}+88=0. \tag{A.6}$$

If we pick the largest again root $\mu_4 \approx 11.9972$ of this equation we can see that indeed condition $0 \leq p_4 - p_3 \leq p_3 - p_2$ is respected (it is equivalent to $0 \leq k \leq 0.204082$ which holds since $k \leq \frac{1}{r+1}$ and $r \geq 4$), so we have computed that for any m

$$p_4^{(m)} \approx 4 - \frac{11.9972}{m+1} \quad \text{and} \quad \mu_4 \approx 11.9972.$$

- $r = 5$ and $m = 5$: At this point we need to modify a little bit our procedure of computing the volumes in the usual recursive way, and consider the case where the new p_5 price is such that $p_3 \leq p_5 \leq p_4$ instead of $p_5 \geq p_4$ (and in fact the even stronger condition that $p_5 - p_4 \leq p_4 - p_3$). This is again a straightforward calculation, since as we argued before, $v(p_1, p_2, p_3, p_4, p_5) = v(p_1, p_2, p_3, p_5, p_5)$ and so

$$v(p_1,p_2,p_3,p_4,p_5)=\int_0^{p_5-p_3} v(p_1,p_2,p_3,p_5-t)\,dt+\int_{p_5-p_3}^{p_3-p_2} v(p_1,p_2,p_5-t,p_5-t)\,dt$$
$$+\int_{p_3-p_2}^{p_2-p_1} v(p_1,p_3-t,p_5-t,p_5-t)\,dt+\int_{p_2-p_1}^{p_1} v(p_2-t,p_3-t,p_5-t,p_5-t)\,dt+\int_{p_1}^{1} dt$$

which equals

$$\frac{1}{120}\left(-5p_1^5+25p_1^4p_5-50p_1^3p_5^2+50p_1^2\left(6p_3^3-18p_3^2p_5+18p_3p_5^2-5p_5^3\right)+25p_1\left(4p_2^4-16p_2^3p_5\right.\right.$$
$$\left.+24p_2^2p_5^2-16p_2\left(3p_3^3-9p_3^2p_5+9p_3p_5^2-2p_5^3\right)+18p_3^4-48p_3^3p_5+36p_3^2p_5^2-3p_5^4\right)-2\left(20p_2^5\right.$$
$$\left.\left.-75p_2^4p_5+100p_2^3p_5^2-50p_2^2\left(3p_3^3-9p_3^2p_5+9p_3p_5^2-2p_5^3\right)+30p_3^5-75p_3^4p_5+50p_3^3p_5^2-2p_5^5-60\right)\right) \tag{A.7}$$

By applying the transformation (4.6), plugging in the already computed values for $\mu_1 = 1$ and $\mu_2 = 2+\sqrt{2}$ and using the fact that μ_3 is the root of Equation (A.4), we get that equation $v(1-\mu_1 k, 2-\mu_2 k, 3-\mu_3 k, 5-\mu_5 k, 5-\mu_5 k) = 5k$ is equivalent to

$$4\mu_5^5 - 225\mu_5^4 + 4350\mu_5^3 + 800\sqrt{2}\mu_5^2 - 34950\mu_5^2 + 900\mu_5\mu_3^2 - 1800\sqrt{2}\mu_5\mu_3 - 3600\mu_5\mu_3 - 14600\sqrt{2}\mu_5 + 121175\mu_5 + 720\sqrt{2}\mu_3^2 - 14220\mu_3^2 + 22680\sqrt{2}\mu_3 + 49680\mu_3 + 41080\sqrt{2} - 161215 = 0 \tag{A.8}$$

If we pick the second largest root $\mu_5 \approx 18.0865$ of this equation we can see that indeed condition $p_3 \leq p_5 \leq p_4$ is respected (it is equivalent to $0.16422 \leq k \leq 0.181994$ which holds since $k = \frac{1}{m+1}$ and $m = 5$), so we have computed that for $m = 5$

$$p_5^{(5)} \approx 5 - \frac{18.0865}{6} = 1.98558 \quad \text{and} \quad \mu_5^{(5)} \approx 18.0865.$$

- $r = 5$ and any $m \geq 6$: We can compute that for every p_5 such that $0 \leq p_5 - p_4 \leq p_4 - p_3$ it is

$$\begin{aligned} v(p_1,p_2,p_3,p_4,p_5) = &\int_0^{p_5-p_4} v(p_1,p_2,p_3,p_4)\,dt + \int_{p_5-p_4}^{p_4-p_3} v(p_1,p_2,p_3,p_5-t)\,dt \\ &+ \int_{p_4-p_3}^{p_3-p_2} v(p_1,p_2,p_4-t,p_5-t)\,dt + \int_{p_3-p_2}^{p_2-p_1} v(p_1,p_3-t,p_4-t,p_5-t)\,dt \\ &+ \int_{p_2-p_1}^{p_1} v(p_2-t,p_3-t,p_4-t,p_5-t)\,dt + \int_{p_1}^{1} dt \end{aligned}$$

which equals

$$\begin{aligned} &\frac{1}{120}\Big(-5p_1^5 + 25p_1^4p_5 + 50p_1^3\left(4p_4^2 - 8p_4p_5 + 3p_5^2\right) + 50p_1^2\Big(6p_3^3 - 18p_3^2p_5 - 18p_3\Big(2p_4^2 \\ &-4p_4p_5 + p_5^2\Big) + 16p_4^3 - 24p_4^2p_5 + 3p_5^3\Big) + 25p_1\Big(4p_2^4 - 16p_2^3p_5 - 24p_2^2\left(3p_4^2 - 6p_4p_5 + 2p_5^2\right) \\ &-16p_2\left(3p_3^3 - 9p_3^2p_5 - 9p_3\left(2p_4^2 - 4p_4p_5 + p_5^2\right) + 6p_4^3 - 9p_4^2p_5 + p_5^3\right) + 18p_3^4 - 48p_3^3p_5 \\ &-36p_3^2\left(2p_4^2 - 4p_4p_5 + p_5^2\right) + 12p_4^4 - 16p_4^3p_5 + p_5^4\Big) - 40p_2^5 + 150p_2^4p_5 + 200p_2^3\Big(3p_4^2 \\ &-6p_4p_5 + 2p_5^2\Big) + 100p_2^2\left(3p_3^3 - 9p_3^2p_5 - 9p_3\left(2p_4^2 - 4p_4p_5 + p_5^2\right) + 6p_4^3 - 9p_4^2p_5 + p_5^3\right) \\ &-60p_3^5 + 150p_3^4p_5 + 200p_3^3p_4^2 - 400p_3^3p_4p_5 + 100p_3^3p_5^2 - 20p_4^5 + 25p_4^4p_5 - p_5^5 + 120\Big) \end{aligned} \tag{A.9}$$

By applying the transformation (4.6), plugging in the already computed values for $\mu_1 = 1$ and $\mu_2 = 2+\sqrt{2}$ and using the fact that μ_3 is the root of Equation (A.4) and μ_4 is the root of (A.6), we get that equation $v(1-\mu_1 k, 2-\mu_2 k, 3-\mu_3 k, 4-$

$\mu_4 k, 5 - \mu_5 k) = 5k$ is equivalent to

$$\begin{aligned}
&\mu_5^5-25\mu_5^4+50\mu_5^3-100\mu_5^2\mu_3^3+900\mu_5^2\mu_3^2-900\mu_5^2\mu_3-800\sqrt{2}\mu_5^2-950\mu_5^2-150\mu_5\mu_3^4+400\mu_5\mu_3^3\mu_4 \\
&+1200\mu_5\mu_3^3-3600\mu_5\mu_3^2\mu_4-900\mu_5\mu_3^2+3600\mu_5\mu_3\mu_4-25\mu_5\mu_4^4+400\mu_5\mu_4^3-600\mu_5\mu_4^2+2400\sqrt{2}\mu_5\mu_4 \\
&+2800\mu_5\mu_4-1600\sqrt{2}\mu_5-2225\mu_5+60\mu_3^5-150\mu_3^4-200\mu_3^3\mu_4^2-400\mu_3^3\mu_4-1900\mu_3^3+1800\mu_3^2\mu_4^2+3600\mu_3^2\mu_4 \\
&-1800\mu_3\mu_4^2-3600\mu_3\mu_4+1800\mu_3+20\mu_4^5-275\mu_4^4-1200\sqrt{2}\mu_4^2-800\mu_4^2-2400\sqrt{2}\mu_4-2800\mu_4+8960\sqrt{2} \\
&+12185=0. \qquad \text{(A.10)}
\end{aligned}$$

If we pick the largest again (real) root $\mu_5 \approx 18.0843$ of this equation we can see that indeed condition $0 \le p_5 - p_4 \le p_4 - p_3$ is respected (it is equivalent to $0 \le k \le 0.16428$ which holds since $k \le \frac{1}{m+1}$ and $m \ge 6$), so we have computed that for any $m \ge 6$

$$p_5^{(m)} \approx 5 - \frac{18.0843}{m+1} \quad \text{and} \quad \mu_5^{(m)} \approx 18.0843.$$

- $\underline{r = 6 \text{ and } m = 6}$: If the new p_6 price is such that $p_4 \le p_6 \le p_5$, similar to the case of $r = m = 5$, we have that $v(p_1, p_2, p_3, p_4, p_5, p_6) = v(p_1, p_2, p_3, p_4, p_6, p_6)$ and so

$$\begin{aligned}
v(p_1,p_2,p_3,p_4,p_5,p_6) = &\int_0^{p_6-p_4} v(p_1,p_2,p_3,p_4,p_6-t)\,dt + \int_{p_6-p_4}^{p_4-p_3} v(p_1,p_2,p_3,p_5-t,p_5-t)\,dt \\
&+ \int_{p_4-p_3}^{p_3-p_2} v(p_1,p_2,p_4-t,p_6-t,p_6-t)\,dt + \int_{p_3-p_2}^{p_2-p_1} v(p_1,p_3-t,p_4-t,p_6-t,p_6-t)\,dt \\
&+ \int_{p_2-p_1}^{p_1} v(p_2-t,p_3-t,p_4-t,p_6-t,p_6-t)\,dt + \int_{p_1}^{1} dt
\end{aligned}$$

which equals

$$\begin{aligned}
&\frac{1}{720}\Big(6p_1^6 - 36p_1^5p_6 + 90p_1^4p_6^2 - 120p_1^3\left(10p_4^3 - 30p_4^2p_6 + 30p_4p_6^2 - 9p_6^3\right) - 90p_1^2\big(10p_3^4 - 40p_3^3p_6 \\
&+60p_3^2p_6^2 - 40p_3\left(3p_4^3 - 9p_4^2p_6 + 9p_4p_6^2 - 2p_6^3\right) + 60p_4^4 - 160p_4^3p_6 + 120p_4^2p_6^2 - 11p_6^4\big) - 36p_1\big(5p_2^5 \\
&-25p_2^4p_6 + 50p_2^3p_6^2 - 50p_2^2\left(6p_4^3 - 18p_4^2p_6 + 18p_4p_6^2 - 5p_6^3\right) - 25p_2\big(4p_3^4 - 16p_3^3p_6 + 24p_3^2p_6^2 \\
&-16p_3\left(3p_4^3 - 9p_4^2p_6 + 9p_4p_6^2 - 2p_6^3\right) + 18p_4^4 - 48p_4^3p_6 + 36p_4^2p_6^2 - 3p_6^4\big) + 2\big(20p_3^5 - 75p_3^4p_6 \\
&+100p_3^3p_6^2 - 50p_3^2\left(3p_4^3 - 9p_4^2p_6 + 9p_4p_6^2 - 2p_6^3\right) + 30p_4^5 - 75p_4^4p_6 + 50p_4^3p_6^2 - 2p_6^5\big)\big) + 5\big(15p_2^6 \\
&-72p_2^5p_6 + 135p_2^4p_6^2 - 120p_2^3\left(6p_4^3 - 18p_4^2p_6 + 18p_4p_6^2 - 5p_6^3\right) - 45p_2^2\big(4p_3^4 - 16p_3^3p_6 + 24p_3^2p_6^2 - \\
&16p_3\left(3p_4^3 - 9p_4^2p_6 + 9p_4p_6^2 - 2p_6^3\right) + 18p_4^4 - 48p_4^3p_6 + 36p_4^2p_6^2 - 3p_6^4\big) + 40p_3^6 - 144p_3^5p_6 + 180p_3^4p_6^2 \\
&-80p_3^3\left(3p_4^3 - 9p_4^2p_6 + 9p_4p_6^2 - 2p_6^3\right) + 30p_4^6 - 72p_4^5p_6 + 45p_4^4p_6^2 - p_6^6 + 144\big)\Big) \qquad \text{(A.11)}
\end{aligned}$$

By applying the transformation (4.6), plugging in the already computed values for $\mu_1 = 1$ and $\mu_2 = 2+\sqrt{2}$ and using the fact that μ_3 is the root of Equation (A.4) and μ_4 is the root of (A.6) , we get that equation $v(1-\mu_1 k, 2-\mu_2 k, 3-\mu_3 k, 4-\mu_4 k, 6-\mu_6 k, 6-\mu_6 k) = 6k$ is equivalent to

$$\begin{aligned}
&\mu_6^6 - 36\mu_6^5 + 270\mu_6^4 - 160\mu_6^3\mu_3^3 + 1440\mu_6^3\mu_3^2 - 1440\mu_6^3\mu_3 - 1200\sqrt{2}\mu_6^3 - 2160\mu_6^3 - 180\mu_6^2\mu_3^4 + 720\mu_6^2\mu_3^3\mu_4 \\
&+2160\mu_6^2\mu_3^3 - 6480\mu_6^2\mu_3^2\mu_4 - 7560\mu_6^2\mu_3^2 + 6480\mu_6^2\mu_3\mu_4 + 6480\mu_6^2\mu_3 - 45\mu_6^2\mu_4^4 + 720\mu_6^2\mu_4^3 - 1080\mu_6^2\mu_4^2 \\
&+4320\sqrt{2}\mu_6^2\mu_4 + 5040\mu_6^2\mu_4 + 4320\sqrt{2}\mu_6^2 + 5760\mu_6^2 + 144\mu_6\mu_3^5 - 720\mu_6\mu_3^3\mu_4^2 - 2880\mu_6\mu_3^3\mu_4 - 8640\mu_6\mu_3^3 \\
&+6480\mu_6\mu_3^2\mu_4^2 + 25920\mu_6\mu_3^2\mu_4 + 12960\mu_6\mu_3^2 - 6480\mu_6\mu_3\mu_4^2 - 25920\mu_6\mu_3\mu_4 - 6480\mu_6\mu_3 + 72\mu_6\mu_4^5 \\
&-900\mu_6\mu_4^4 - 1440\mu_6\mu_4^3 - 4320\sqrt{2}\mu_6\mu_4^2 - 720\mu_6\mu_4^2 - 17280\sqrt{2}\mu_6\mu_4 - 20160\mu_6\mu_4 + 8928\sqrt{2}\mu_6 + 13104\mu_6 \\
&-40\mu_3^6 - 144\mu_3^5 + 1440\mu_3^4 + 240\mu_3^3\mu_4^3 + 1440\mu_3^3\mu_4^2 + 2880\mu_3^3\mu_4 + 10560\mu_3^3 - 2160\mu_3^2\mu_4^3 - 12960\mu_3^2\mu_4^2 \\
&-25920\mu_3^2\mu_4 - 7560\mu_3^2 + 2160\mu_3\mu_4^3 + 12960\mu_3\mu_4^2 + 25920\mu_3\mu_4 - 2160\mu_3 - 30\mu_4^6 + 288\mu_4^5 + 1440\mu_4^4 \\
&+1440\sqrt{2}\mu_4^3 + 1680\mu_4^3 + 8640\sqrt{2}\mu_4^2 + 5760\mu_4^2 + 17280\sqrt{2}\mu_4 + 20160\mu_4 - 42048\sqrt{2} - 58344 = 0
\end{aligned} \tag{A.12}$$

If we pick the second largest root $\mu_6 \approx 25.3585$ of this equation we can see that indeed condition $p_4 \leq p_6 \leq p_5$ is respected (it is equivalent to $0.137473 \leq k \leq 0.149686$ which holds since $k = \frac{1}{m+1}$ and $m = 6$), so we have computed that for $m = 6$

$$p_6^{(6)} \approx 6 - \frac{25.3585}{7} = 2.37736 \quad \text{and} \quad \mu_6^{(6)} \approx 25.3585.$$

A.2 An Alternative Explicit Dual Solution for Two Uniform Goods

In this section we provide an alternative optimal dual solution to these given previously in Chapter 4 and Section 5.6, for the case of two uniformly i.i.d. goods over the unit interval $[0, 1]$. In the former, we proved the *existence* of a proper dual solution that matched, through approximate complementarity, the primal one given by the SJA selling mechanism, while in the latter we *explicitly constructed* a dual solution that was shown to be optimal by exact complementarity. The reason we choose to present here one more dual solution is that it is rather different in character and we believe it demonstrates certain subtle features of the duality framework that are of interest; first, the two components (z_1, z_2) of our dual solution here are highly *non-symmetric*, meaning that $z_1(x_1, x_2) \neq z_2(x_2, x_1)$. The solutions in the main part of this thesis avoid this, for reasons of simplicity and generality, however we think it is important to make clear that symmetry is not a requirement and, furthermore, the optimal dual value can be achieved by various, completely different, feasible dual solutions. Secondly, the solution will demonstrate a degree of *non-continuity*: remember that we demand our dual solutions z_j to be absolutely continuous with respect to their critical j-th component, however it is perfectly acceptable to have discontinuities with respect to their other coordinates as long as the functions remain integrable in the domain (see

Footnote 3). This was not made explicit via the solution given in Section 5.6, which was quite smooth. Finally, here we don't utilize complementarity, but simple weak duality to show optimality: we give a dual solution depending on a small parameter ε, prove that it gives rise to a dual value which is $O(\varepsilon)$ close to the primal one, and take $\varepsilon \to 0$ to establish optimality.

First, recall that we already know that in our setting of two uniform i.i.d. goods the optimal selling mechanism is the one given by the utility function

$$u(x_1, x_2) = \max_{\mathbf{x} \in I^2} \{0, x_1 - \alpha, x_2 - \alpha, x_1 + x_2 - \beta\}, \tag{A.13}$$

where parameters α and β are selected to be

$$\alpha = \frac{2}{3} \quad \text{and} \quad \beta = \frac{4 - \sqrt{2}}{3},$$

i.e. β being the a root of $9x^2 - 24x + 14$ in $[0, 2\alpha]$. You can see a graphical representation of the allocation function and payment p of this mechanism in Figure A.1. It is the symmetric deterministic one that offers either each one of the items for a price of $\frac{2}{3}$ or their full bundle for a price of $\frac{4-\sqrt{2}}{3} \approx 0.862$.

Next, we are going to construct the family of our dual feasible solutions $z_1, z_2 : I^2 \longrightarrow \mathbb{R}_{\geq 0}$. Fix a parameter $\varepsilon > 0$ with $\alpha \leq 1 - \varepsilon < 1$ and define:

$$z_1(x_1, x_2) = \begin{cases} \frac{3}{2\varepsilon} x_1^2 + \frac{2-3\beta}{\varepsilon} x_1 + \frac{(\alpha-\beta)(4-3\alpha-3\beta)}{2\varepsilon}, & \beta - \alpha \leq x_1 \leq \alpha \ \wedge \ 1 - \varepsilon \leq x_2 \leq 1, \\ -\frac{1-3\varepsilon}{\varepsilon} x_1 + \frac{3\beta^2 - 4\beta + 10\alpha - 6\alpha\beta - 6\alpha\varepsilon}{2\varepsilon}, & \alpha < x_1 \leq 1 \ \wedge \ 1 - \varepsilon < x_2 \leq 1, \\ 3(x_1 - \alpha), & \alpha < x_1 \leq 1 \ \wedge \ 0 \leq x_2 \leq 1 - \varepsilon, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$z_2(x_1, x_2) = \begin{cases} 3(x_2 - \alpha), & 0 \leq x_1 \leq \beta - \alpha \ \wedge \ \alpha \leq x_2 \leq 1, \\ 3(x_1 + x_2 - \beta), & \alpha - \beta < x_1 \leq \alpha \ \wedge \ \beta - x_1 \leq x_2 \leq 1 - \varepsilon, \\ \frac{3(x_1 - \beta - \varepsilon) + 2}{\varepsilon}(1 - x_2) + 1, & \alpha - \beta < x_1 \leq \alpha \ \wedge \ 1 - \varepsilon < x_2 \leq 1, \\ \frac{x_2 + \varepsilon - 1}{\varepsilon}, & \alpha < x_1 \leq 1 \ \wedge \ 1 - \varepsilon < x_2 \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $z_1(0, x_2) = 0$ and $z_2(x_1, 0) = 0$ for all $x_1, x_2 \in I$, trivially by definition, satisfying the first dual constraint (6.1). Then, we compute

$$z_1(1, x_2) = \begin{cases} 3(1 - \alpha) + \frac{3\beta^2 - 4\beta + 10\alpha - 6\alpha\beta - 2}{2\varepsilon}, & 1 - \varepsilon < x_2 \leq 1, \\ 3(1 - \alpha), & 0 \leq x_2 \leq 1 - \varepsilon, \end{cases}$$

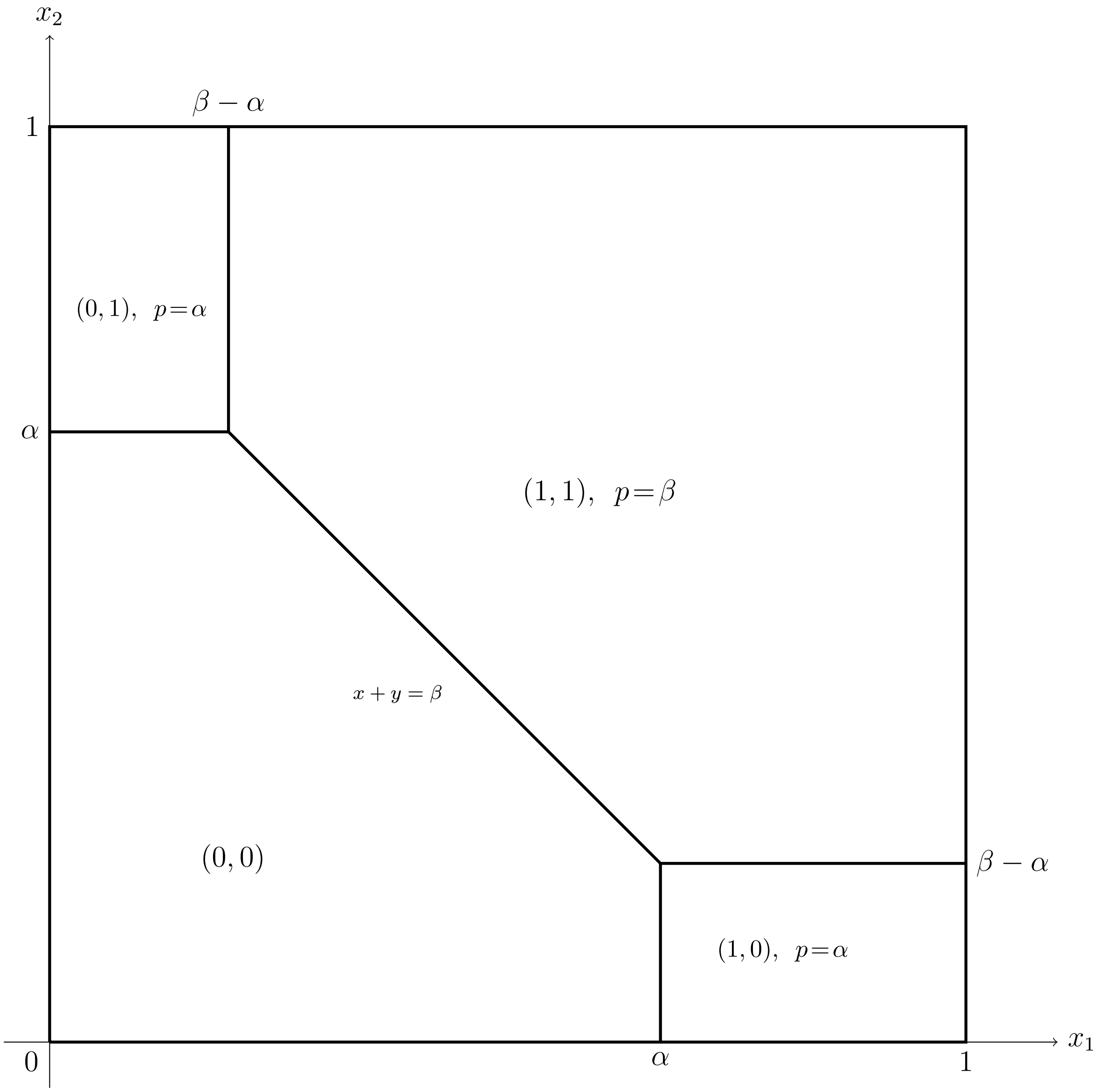


Figure A.1: The optimal mechanism for selling two uniform i.i.d. goods over $[0,1]$.

and

$$z_2(x_1,1) = \begin{cases} 3(1-\alpha), & 0 \leq x_1 \leq \beta - \alpha, \\ 1, & \alpha - \beta < x_1 \leq 1, \end{cases}$$

for all $x_1, x_2 \in I$. Thus, since $\alpha = \frac{2}{3}$ and β is the root of $9x^2 - 24x + 14$ we ensure that $z_1(1,x_2) = 1$ and $z_2(x_1,1) = 1$ for all $x_1, x_2 \in I$, satisfying the second dual constraint (6.2). Also, it is not difficult to check that z_1 and z_2 take only nonnegative values. Furthermore, z_1 is absolutely continuous with respect to its first coordinate x_1 and only discontinuous at the line segment $x_2 = 1 - \varepsilon \ \wedge \ \beta - \alpha < x_2 < 1$ which is a set of measure 0 within I^2, thus z_1 is integrable in I^2. Similarly, z_2 is absolutely continuous with respect to its second coordinate x_2 and only discontinuous at the line segment $x_1 = \alpha \ \wedge \ \beta - \alpha < x_1 < 1$. An illustration of the critical regions of z_1 and z_2 and their values is given in Figures A.2 and A.3, respectively.

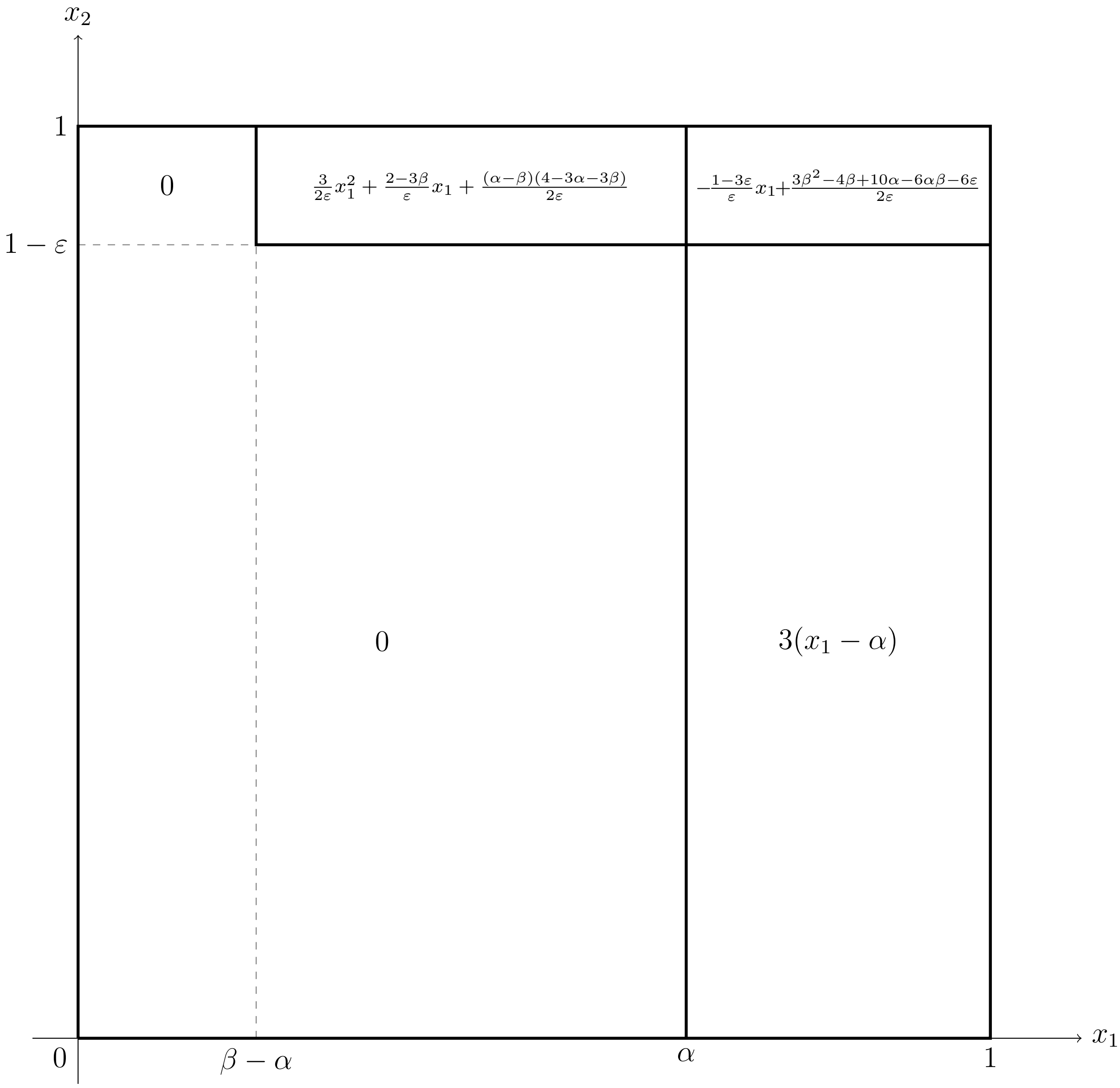


Figure A.2: The critical regions of the dual solution z_1

In addition, z_1 and z_2 are almost everywhere differentiable in I^2 with:

$$\frac{\partial z_1(x_1,x_2)}{\partial x_1} = \begin{cases} \frac{3}{\varepsilon}x_1 + \frac{2-3\beta}{\varepsilon}, & \beta-\alpha \le x_1 \le \alpha \ \wedge \ 1-\varepsilon \le x_2 \le 1, \\ 3-\frac{1}{\varepsilon}, & \alpha < x_1 \le 1 \ \wedge \ 1-\varepsilon < x_2 \le 1, \\ 3, & \alpha < x_1 \le 1 \ \wedge \ 0 \le x_2 \le 1-\varepsilon, \\ 0, & \text{otherwise,} \end{cases}$$

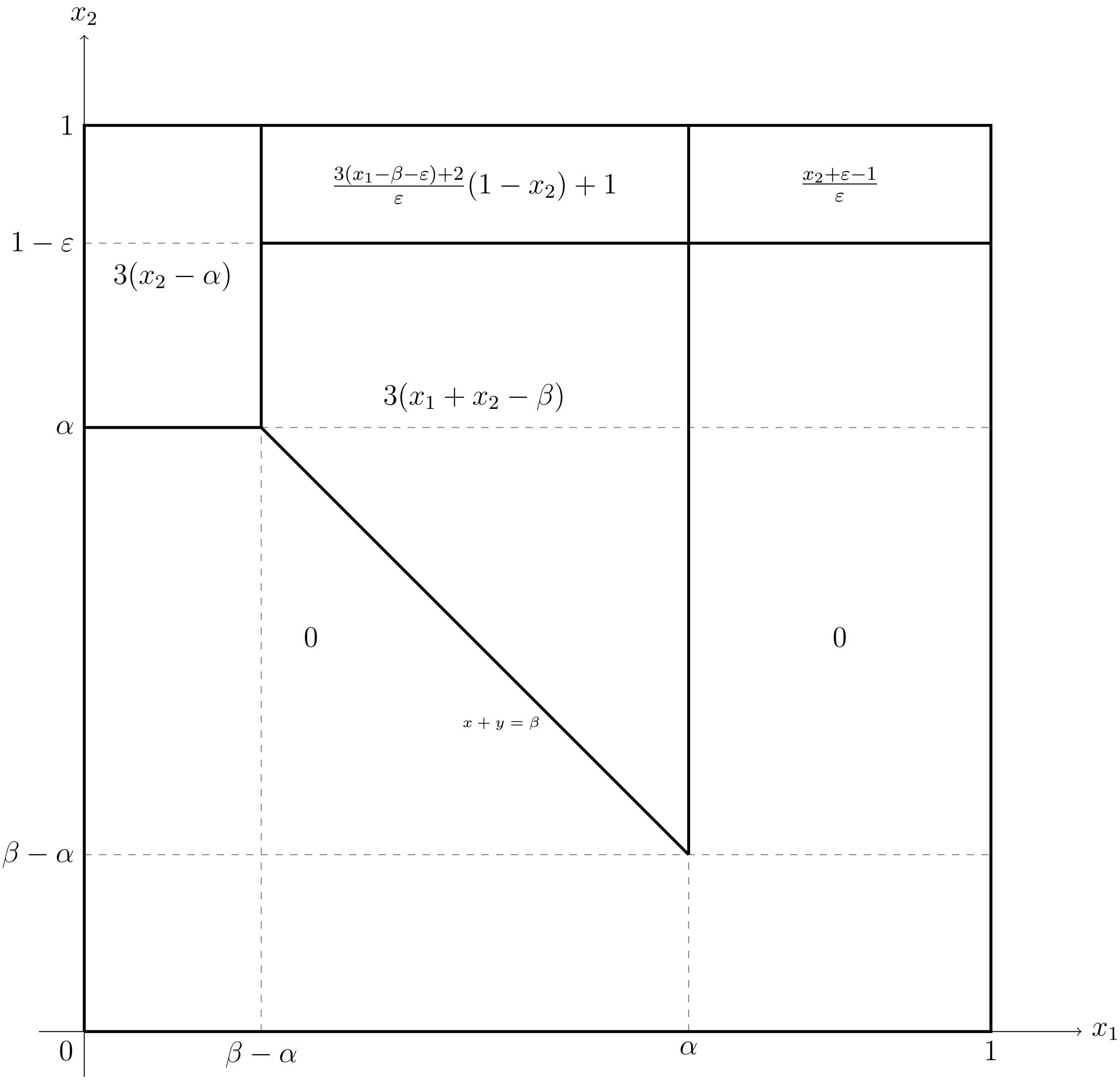


Figure A.3: The critical regions of the dual solution z_2

and

$$\frac{\partial z_2(x_1,x_2)}{x_2} = \begin{cases} 3, & 0 \leq x_1 \leq \beta-\alpha \ \wedge \ \alpha \leq x_2 \leq 1, \\ 3, & \alpha-\beta < x_1 \leq \alpha \ \wedge \ \beta - x_1 \leq x_2 \leq 1-\varepsilon, \\ -\frac{3(x_1-\beta-\varepsilon)+2}{\varepsilon}, & \alpha-\beta < x_1 \leq \alpha \ \wedge \ 1-\varepsilon < x_2 \leq 1, \\ \frac{1}{\varepsilon}, & \alpha < x_1 \leq 1 \ \wedge \ 1-\varepsilon < x_2 \leq 1, \\ 0, & \text{otherwise}, \end{cases}$$

so by summing up in the various subregions of I^2 we immediately get that also condition (6.3) is satisfied. Finally, it remains to compute the dual value (6.4) for the current choice of z_1 and z_2 and prove that it provides a *tight* upper bound on the revenue of the deterministic auction induced by the utility function in (A.13), which is (consult

Figure A.1)

$$2\int_{\alpha}^{1}\int_{0}^{\beta-\alpha}\alpha\,dx_2\,dx_1+\int_{\beta-\alpha}^{1}\int_{\alpha}^{1}\beta\,dx_2\,dx_1+\int_{\beta-\alpha}^{\alpha}\int_{\beta-x_2}^{1}\beta\,dx_1\,dx_2$$

which equals

$$2\alpha^3-\alpha^2(3\beta+2)+4\alpha\beta+\frac{1}{2}\beta\left(\beta^2-4\beta+2\right). \tag{A.14}$$

We now compute the dual objective value (consult Figures A.2 and A.3):

$$\begin{aligned}\int_0^1\int_0^1 z_1(x_1,x_2) &= \int_{1-\varepsilon}^{1}\int_{\beta-\alpha}^{\alpha}\frac{3}{2\varepsilon}x_1^2+\frac{2-3\beta}{\varepsilon}x_1+\frac{(\alpha-\beta)(4-3\alpha-3\beta)}{2\varepsilon}\,dx_1\,dx_2\\ &+\int_{1-\varepsilon}^{1}\int_{\alpha}^{1}-\frac{1-3\varepsilon}{\varepsilon}x_1+\frac{3\beta^2-4\beta+10\alpha-6\alpha\beta-6\varepsilon}{2\varepsilon}\,dx_1\,dx_2+\int_0^{1-\varepsilon}\int_{\alpha}^{1}3(x_1-\alpha)\,dx_1\,dx_2\\ &= -2\alpha^3+3\alpha^2\beta+\alpha^2-5\alpha\beta+2\alpha-\frac{\beta^3}{2}+\frac{5\beta^2}{2}-2\beta+1\end{aligned}$$

and

$$\begin{aligned}\int_0^1\int_0^1 z_2(x_1,x_2) &= \int_{\alpha}^{1}\int_{0}^{\beta-\alpha}3(x_2-\alpha)\,dx_2\,dx_1+\int_{\beta-\alpha}^{\alpha}\int_{\beta-x_1}^{1-\varepsilon}3(x_1+x_2-\beta)\,dx_2\,dx_1\\ &+\int_{\beta-\alpha}^{\alpha}\int_{1-\varepsilon}^{1}\frac{3(x_1-\beta-\varepsilon)+2}{\varepsilon}(1-x_2)+1\,dx_2\,dx_1+\int_{\alpha}^{1}\int_{1-\varepsilon}^{1}\frac{x_2+\varepsilon-1}{\varepsilon}\,dx_2\,dx_1\\ &= \frac{1}{4}\left(-2\alpha^3+12\alpha^2+6\alpha\beta^2-24\alpha\beta+6\alpha-2\beta^3+6\beta^2\right)+\frac{1}{4}\varepsilon\left(2\alpha(3\beta-5)-3\beta^2+4\beta+2\right)\end{aligned}$$

By summing up and letting $\varepsilon\to 0$, objective (6.4) is

$$\begin{aligned}\int_0^1\int_0^1 z_1(x_1,x_2)+z_2(x_1,x_2)\,dx_1\,dx_2 &= \frac{1}{2}\left(-5\alpha^3+\alpha^2(6\beta+8)+\alpha\left(3\beta^2-22\beta+7\right)-2\beta^3+8\beta^2-4\beta+2\right)\\ &+O(\varepsilon).\end{aligned} \tag{A.15}$$

By taking expressions (A.14) and (A.15) to be equal we get equation

$$9\alpha^3+\alpha\left(-3\beta^2+30\beta-7\right)+3\beta^3+6\beta=12\alpha^2(\beta+1)+12\beta^2+2,$$

which for $\alpha=\frac{2}{3}$ becomes

$$(\beta-2)(9\beta^2-24\beta+14)=0,$$

thus being satisfied by our initial choice of α and β.

A.3 General k-Regularity

In this section we present a theory of *regularity* for the distributional priors of the bids, which generalizes the traditional notion of regularity used by Myerson [58] (see Definition 2.6). This is going to allow us to generalize the construction we used in

Section 6.1 to prove good upper bounds for any number m of uniformly distributed goods, to the case of general k-*regular* distributions in Appendix A.3.2. We choose to make a full discussion and exposition of this idea in this appendix, since we believe it may contain ideas to help with the development of non-trivial optimality results for general distributions and multiple items and players, which is the main challenging open problem in the field.

A.3.1 Regular distributions

Now we define our generalized notion of k-regularity, inspired by the techniques in Section 6.1:

Definition A.1 (k-Regularity)**.** Let m, k be positive integers such that $1 \leq k \leq m$. A distribution F on I will be called k-*regular*, if the *order-k virtual valuation* induced, defined by

$$_k\tilde{v} = {}_k\tilde{v}(x) \equiv x - d_k \frac{1 - F(x)}{f(x)},$$

is a strictly[1] increasing function, where

$$d_k \equiv \frac{m+1}{k} - 1.$$

In case of k-regular distributions we will call k-*regular root*, denoted by $_k\tilde{x}^*$, the (unique) root of $_k\tilde{v}$ in I.

Definition A.2. Let m, k be positive integers such that $1 \leq k \leq m$. We define the *positive projection* of the order-k virtual valuation $_k\tilde{v}$ as the function

$$_k\tilde{V} \equiv \max\{0, {}_k\tilde{v}\} = \max\left\{0, x - d_k \frac{1 - F(x)}{f(x)}\right\}, \qquad x \in I.$$

In case of k-regular distributions this can be expressed as

$$_k\tilde{V}(x) = \begin{cases} 0, & \text{if } 0 \leq x < {}_k\tilde{x}^*, \\ x - d_k \frac{1-F(x)}{f(x)}, & \text{if } {}_k\tilde{x}^* \leq x \leq 1. \end{cases}$$

We extend the above definitions to the special case of $k = 0$ by defining $d_0 \equiv 1$. In this case, we will call 0-regular distributions simply *regular*, since this coincides with the standard definition of regularity in [58], and drop the subscript k.

An important example of k-regular (and regular) distribution is the uniform distri-

[1] Weak monotonicity can easily be incorporated in our exposition, however we choose to not allow it for reasons of simplicity when defining the notion of k-*regular root* (see below). One can redefine this to be the leftmost root of the virtual valuation and appropriately adapt the analysis.

bution U, for which we have

$$_k\tilde{v}(x) = \frac{m+1}{k}x + 1 - \frac{m+1}{k} \quad \text{and} \quad {_k\tilde{x}^*} = 1 - \frac{k}{m+1} \tag{A.16}$$

for $k \geq 1$ and $\tilde{v}(x) = 2x - 1$, $\tilde{x}^* = \frac{1}{2}$.

A.3.2 Bounds on Optimal Revenue

No we use weak duality to get upper-bound formulas for the optimal revenue in two settings of gradually increasing specialization: first for general, m-regular (not necessarily identical) distributional priors (Theorem A.2) and then for i.i.d. k-regular (for all $k \in [m]$) ones (Theorem A.3). For ease of reference, let us restate the Weak Duality Lemma 3.1 (see also the dual Program (3.5)) for our particular case of independent valuations over I. The valuation of item j is drawn from distribution F_j, and their product is denoted by $F = \prod_{j=1}^m F_j$.

Theorem A.1 (Weak Duality for independent goods over I)**.** *The dual constraints for a single buyer and m independent goods over I^m become:*

$$z_j(0, \mathbf{x}_{-j}) = 0, \qquad \textit{for all } j \in [m], \tag{A.17}$$

$$z_j(\mathbf{x}_{-j}, 1) \geq f_j(1) \prod_{k \neq j} f_k(x_k), \qquad \textit{for all } j \in [m], \tag{A.18}$$

$$\sum_{j=1}^m \frac{\partial z_j(\mathbf{x})}{\partial x_j} \leq \prod_{j=1}^m f_j(x_j) + \sum_{j=1}^m \left((x_j f_j(x_j))' \prod_{k \neq j} f_k(x_k) \right), \tag{A.19}$$

and the dual objective upper-bounds optimal revenue:

$$\text{Rev}\,(F) \leq \sum_{j=1}^m \int_{I^m} z_j(\mathbf{x})\, d\mathbf{x}. \tag{A.20}$$

Theorem A.2. *Any selling mechanism for m independent goods has an expected revenue of at most*

$$\text{Rev}(F) \leq \sum_{j=1}^m \mathbb{E}\left[_m\tilde{V}_j\right]$$

where $_mV_j$'s are the positive projections of the order-m virtual valuations (see Definition A.2).

In case of m-regular distributions this becomes

$$\sum_{j=1}^m \int_{_m\tilde{x}_j^*}^1 x f_j(x) - \frac{1 - F_j(x)}{m}\, dx,$$

where $_m\tilde{x}^*$ is the m-regular root (see Definitions A.1 and A.2).

Proof. We are going to construct feasible z_j's, $j = 1, 2, \ldots, m$, to plug them into

Theorem A.1. Set

$$z_j(\mathbf{x}) = \prod_{j=1}^{m} f_j(x_j)\left(\max\left\{0, x - \frac{1-F_j(x)}{mf_j(x)}\right\}\right) = \prod_{j=1}^{m} f_j(x_j)_m\tilde{V}_j(x_j).$$

It is easy to check that this choice of z_j's satisfies conditions (A.19)–(A.18), thus giving an upper bound on the optimal expected revenue of (see equation (A.20))

$$\begin{aligned}
\int_{[0,1]^m} \sum_{j=1}^{m} z_j(\mathbf{x})\,d\mathbf{x} &= \sum_{j=1}^{m} \int_{[0,1]^m} \prod_{j=1}^{m} f_j(x_j)_m\tilde{V}_j(x_j)\,d\mathbf{x} \\
&= \sum_{j=1}^{m} \left(\int_{[0,1]^{m-1}} \prod_{k\neq j} f_k(x_k)\,d\mathbf{x}_{-j} \int_0^1 f_j(x_j)_m\tilde{V}_j(x_j)\,dx_j\right) \\
&= \sum_{j=1}^{m} \int_0^1 {}_m\tilde{V}_j(x_j) f_j(x_j)\,dx_j \qquad \text{(A.21)} \\
&= \sum_{j=1}^{m} \mathbb{E}\left[{}_m\tilde{V}_j\right].
\end{aligned}$$

In case of m-regular distributions, equation (A.21) gives

$$\sum_{j=1}^{m} \int_0^1 {}_m\tilde{V}_j(x_j) f_j(x_j)\,dx_j = \sum_{j=1}^{m} \int_{{}_m\tilde{x}_j^*}^1 {}_m\tilde{V}_j(x) f_j(x)\,dx = \sum_{j=1}^{m} \int_{{}_m\tilde{x}_j^*}^1 x f_j(x) - \frac{1-F_j(x)}{m}\,dx.$$

□

For example, for the case of uniform distributions Theorem A.2 would give an upper bound of (see equation (A.16)) $\textsc{Rev}(\mathcal{U}^m) \le m\int_{\frac{1}{m+1}}^{1} \frac{m+1}{m}x - \frac{1}{m}\,dx = \frac{m^2}{2(1+m)}$. This slightly improves the trivial upper bound of $\frac{m}{2}$ taken by the IR constraint[2] but it is still not as good as the bound from Theorem 6.2. That improved theorem is a specialization of the general dual-construction technique given by the following:

Theorem A.3. *Any selling mechanism for m i.i.d. goods following a regular and k-regular (for all $k = 1, 2, \ldots, m$) distribution F has an expected revenue of at most:*

$$\textsc{Rev}(F^m) \le F({}_m\tilde{x}^*)^{m-1} \sum_{k=1}^{m} \binom{m}{k} k \left(\frac{1}{F({}_m\tilde{x}^*)} - 1\right)^{k-1} \int_{{}_k\tilde{x}^*}^1 x f(x) - d_k(1-F(x))\,dx$$

Proof. (As a warm-up for this general proof, the reader can find a proof of this theorem for the special case of uniform i.i.d. distributions given in Section 6.1 of the thesis.) The fact that the goods are identical is only used at the last part of the computation of the actual dual objective, so trying to leave our construction as general as possible, until then we will consider that the valuation of item j comes from distribution F_j, and later we will replace $F = F_1 = \cdots = F_m$.

[2] See the discussion after Theorem 6.2.

Again, we will construct feasible solutions z_j, for Theorem A.1. Fix regular and k-regular (for all orders $k = 1, 2, \ldots, m$) valuation distributions $F_1, F_2, \ldots, F_m$. For every node $\mathbf{v} = (v_1, v_2, \ldots, v_m) \in \mathcal{I}_m$ of the m-dimensional unit hypercube define $L_{\mathbf{x}}$ to be the following subspace of I^m:

$$L_{\mathbf{v}} = \{(x_1, x_2, \ldots, x_m) \in I^m \mid x_j \in [0, {}_m\tilde{x}^*] \text{ if } v_j = 0 \text{ and } x_j \in ({}_m\tilde{x}^*, 1] \text{ if } v_j = 1, \quad j \in [m]\}$$

Due to the regularity of the distributions, it is a simple observation that $L_{\mathbf{v}}$'s form a valid partition of I^m, i.e.

$$\mathbf{v}, \mathbf{v}' \in \mathcal{I}_m \wedge \mathbf{v} \neq \mathbf{v}' \Longrightarrow L_{\mathbf{v}} \cap L_{\mathbf{v}'} = \emptyset \quad \text{and} \quad \bigcup_{\mathbf{v} \in \mathcal{I}_m} L_{\mathbf{v}} = I^m$$

Fix some item $j \in [m]$ and a subspace $L_{\mathbf{v}} \subseteq I^m$ (by fixing a $\mathbf{v} = (v_1, v_2, \ldots, v_m) \in \mathcal{I}_m$) and define $z_j : L_{\mathbf{v}} \longrightarrow \mathbb{R}_{\geq 0}$ as follows:

- If $v_j = 0$, set $z_j(\mathbf{x}) = 0$ for all $\mathbf{x} \in L_{\mathbf{v}}$.
- Otherwise, i.e. if $v_j = 1$, set

$$z_j(\mathbf{x}) = \prod_{j=1}^{m} f_j(x_j)_k\tilde{V}_j(x_j) = \begin{cases} 0, & \text{if } {}_m\tilde{x}_j^* \leq x_j < {}_k\tilde{x}_j^*, \\ \prod_{l \neq j} f_l(x_l)\left[x_j f_j(x_j) - d_k(1 - F_j(x))\right], & \text{if } {}_k\tilde{x}_j^* \leq x_j \leq 1. \end{cases}$$

for all $\mathbf{x} \in L_{\mathbf{v}}$, where

$$k = k(\mathbf{v}) = \sum_{j=1}^{m} v_j.$$

By this construction, and by letting $\mathbf{v}$ range over $\mathcal{I}_m$, we have a well defined function $z_j : [0,1]^m \longrightarrow \mathbb{R}_{\geq 0}$. Each $\mathbf{x} \in [0,1]^m$ belongs to a unique partition $L_{\mathbf{v}}$ (corresponding to a *unique* $\mathbf{v} = \mathbf{v}(\mathbf{x})$), thus also well defining $k = k(\mathbf{x})$. This is because ${}_k\tilde{x}_j^*$'s are "well-ordered", i.e.

$${}_m\tilde{x}_j^* \leq {}_{k+1}\tilde{x}_j^* \leq {}_k\tilde{x}_j^* \leq {}_1\tilde{x}_j^* \quad \text{for all } k = 1, 2, \ldots, m-1,$$

since the order k virtual valuations ${}_k\tilde{v}_j$ are non-decreasing functions with ${}_k\tilde{v}_j \leq {}_{k+1}\tilde{v}_j$ (because $d_k \leq d_{k+1}$), for all $k = 1, 2, \ldots, m-1$. So, the above definition can be written more compactly as

$$z_j(\mathbf{x}) = \prod_{j=1}^{m} f_j(x_j)_k\tilde{V}_j(x_j) = \begin{cases} 0, & \text{if } 0 \leq x_j < {}_k\tilde{x}_j^*, \\ \prod_{l \neq j} f_l(x_l)\left[x_j f_j(x_j) - d_k(1 - F_j(x))\right], & \text{if } {}_k\tilde{x}_j^* \leq x_j \leq 1. \end{cases}$$

It is easy to check, directly from this definition, that

$$z_j(0, x_{-j}) = 0 \quad \text{and} \quad z_j(1, x_{-j}) = f_j(1) \prod_{k \neq j} f_k(x_k) \tag{A.22}$$

for all $j \in [m]$ and $x_{-j} \in I^{m-1}$, and also that

$$\frac{\partial z_j(\mathbf{x})}{\partial x_j} = \begin{cases} 0, & \text{if } 0 < x_j \le {}_k\tilde{x}_j^*, \\ \prod_{l \neq j} f_l(x_l) \left[(x_j f_j(x_j))' + d_k f_j(x_j) \right], & \text{if } {}_k\tilde{x}_j^* < x_j \le 1 \end{cases}$$
$$= \begin{cases} 0, & \text{if } 0 < x_j \le {}_k\tilde{x}_j^*, \\ \prod_{l \neq j} f_l(x_l) \left[x_j f_j'(x_j) + (d_k + 1) f_j(x_j) \right], & \text{if } {}_k\tilde{x}_j^* < x_j \le 1 \end{cases}$$
$$= \begin{cases} 0, & \text{if } 0 < x_j \le {}_k\tilde{x}_j^*, \\ \frac{m+1}{k} \prod_{l=1}^m f_l(x_l) + x_j f_j'(x_j) \prod_{l \neq j} f_l(x_l), & \text{if } {}_k\tilde{x}_j^* < x_j \le 1. \end{cases} \tag{A.23}$$

Furthermore, if we fix some $\mathbf{x} \in I^m$ (and thus also fix the corresponding, well-defined, $\mathbf{v} = \mathbf{v}(\mathbf{x}) \in \mathcal{I}_m$ and $k = \sum_{j=1}^m \mathbf{v}_j$), we see from property (A.23) above that

$$\begin{aligned} \sum_{j=1}^m \frac{\partial z_j(\mathbf{x})}{\partial x_j} &\le \sum_{j=1}^m v_j \frac{m+1}{k} \prod_{l=1}^m f_l(x_l) + \sum_{j=1}^m x_j f_j'(x_j) \prod_{l \neq j} f_l(x_l) \\ &= \frac{m+1}{k} \sum_{j=1}^m v_j \prod_{l=1}^m f_l(x_l) + \sum_{j=1}^m x_j f_j'(x_j) \prod_{l \neq j} f_l(x_l) \\ &= \frac{m+1}{k} k \prod_{l=1}^m f_l(x_l) + \sum_{j=1}^m x_j f_j'(x_j) \prod_{l \neq j} f_l(x_l) \\ &= \prod_{j=1}^m f_j(x_j) + \sum_{j=1}^m \left(f_j(x_j) + x_j f_j'(x_j) \right) \prod_{l \neq j} f_l(x_l) \\ &= \prod_{l=1}^m f_l(x_l) + \sum_{j=1}^m (x_j f_j(x_j))' \prod_{l \neq j} f_l(x_l). \end{aligned} \tag{A.24}$$

But now (A.22) and (A.24) are exactly properties (A.17), (A.18) and (A.19) of Theorem A.1.

The last remaining step of our proof is to evaluate the dual objective value. Assuming identical distributions this is:

$$\begin{aligned} \int_{I^m} \sum_{j=1}^m z_j(\mathbf{x})\, d\mathbf{x} &= \sum_{\mathbf{v} \in \mathcal{I}_m} \int_{L_\mathbf{v}} \sum_{j=1}^m z_j(\mathbf{x})\, d\mathbf{x} \\ &= \sum_{\mathbf{v} \in \mathcal{I}_m} \int_{L_\mathbf{v}} \sum_{j:\mathbf{v}_j=1} z_j(\mathbf{x})\, d\mathbf{x} \\ &= \sum_{\kappa=1}^m \sum_{\mathbf{v}:k(\mathbf{v})=\kappa} \int_{L_\mathbf{v}} \sum_{j:\mathbf{v}_j=1} z_j(\mathbf{x})\, d\mathbf{x} \\ &= \sum_{\kappa=1}^m \binom{m}{\kappa} \int_0^{m\tilde{x}^*} \cdots \int_0^{m\tilde{x}^*} \int_{k\tilde{x}^*}^1 \cdots \int_{k\tilde{x}^*}^1 \sum_{j:\mathbf{v}_j=1} z_j(\mathbf{x})\, d\mathbf{x} \\ &= \sum_{k=1}^m \binom{m}{k} \int_0^{m\tilde{x}^*} \cdots \int_0^{m\tilde{x}^*} \int_{k\tilde{x}^*}^1 \cdots \int_{k\tilde{x}^*}^1 \sum_{j:\mathbf{v}_j=1} \prod_{l \neq j} f_l(x_l) \left(x_j f(x_j) - d_k(1 - F(x)) \right) d\mathbf{x} \\ &= \sum_{k=1}^m \binom{m}{k} \left(\int_0^{m\tilde{x}^*} f(x)\, dx \right)^{m-k} \sum_{j:\mathbf{v}_j=1} \int_{k\tilde{x}^*}^1 \cdots \int_{k\tilde{x}^*}^1 \prod_{l \neq j} f_l(x_l) \left(x_j f(x_j) - d_k(1 - F(x)) \right) d\mathbf{x} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{m} \binom{m}{k} \left(\int_0^{{}_m\tilde{x}^*} f(x)\,dx \right)^{m-k} k \left(\int_{{}_m\tilde{x}^*}^1 f(x)\,dx \right)^{k-1} \int_{{}_k\tilde{x}^*}^1 x f(x) - d_k(1-F(x))\,dx \\
&= \sum_{k=1}^{m} \binom{m}{k} F({}_m\tilde{x}^*)^{m-k} k \left(1 - F({}_m\tilde{x}^*)\right)^{k-1} \int_{{}_k\tilde{x}^*}^1 x f(x) - d_k(1-F(x))\,dx \\
&= F({}_m\tilde{x}^*)^{m-1} \sum_{k=1}^{m} \binom{m}{k} F({}_m\tilde{x}^*)^{1-k} k \left(1 - F({}_m\tilde{x}^*)\right)^{k-1} \int_{{}_k\tilde{x}^*}^1 x f(x) - d_k(1-F(x))\,dx \\
&= F({}_m\tilde{x}^*)^{m-1} \sum_{k=1}^{m} \binom{m}{k} k \left(\frac{1}{F({}_m\tilde{x}^*)} - 1 \right)^{k-1} \int_{{}_k\tilde{x}^*}^1 x f(x) - d_k(1-F(x))\,dx.
\end{aligned}
$$

□

A.4 An Explicit, Almost Optimal Dual Solution for Two Exponential Goods

In this section we present an almost optimal, 1.0033–approximate selling mechanism for the case of two goods with valuations following independent, but nonidentical, exponential distributions $F_1(x) = 1-e^{\lambda_1 x}$, $F_2(x) = 1-e^{\lambda_2 x}$, where without loss we assume $\lambda_1 > \lambda_2 > 0$. Notice that if $\lambda_1 = \lambda_2$ we already know that deterministically selling in a full bundle is optimal, from Theorem 6.6. The mechanism is a randomized one that, depending one the value of the ratio $\frac{\lambda_2}{\lambda_1}$ of the distribution parameters, offers either the full bundle or the lighter-tail item with probability 1 and the other with probability $\frac{\lambda_2}{\lambda_1}$ (see Definition A.3). This second component is essentially the PROPORTIONAL (see Definition 6.1) we designed already in Section 6.3.2. We must mention here that our actual results in this section are now obsolete: Daskalakis et al. [25] have shown that this auction is indeed *exactly* optimal. However, our analysis might be of interest to the reader, since it is being done through the use of an *explicit, closed-form* dual solution rather than an existential one like in [25] and, furthermore, the upper bound on the optimal revenue is acquired in a straightforward way via simple weak duality rather than complementarity. In fact, due to the nature of the problem, as we have discussed before throughout this thesis, it might be the case that *exactly tight* closed-form dual solutions cannot be found. Also, the presentation here is more "elementary", and since it shares many points with the more general case of arbitrarily many exponential goods of Section 6.2, it can provide some deeper intuition and illuminate better some technical points. Finally, an interesting feature of our analysis in this section is that the components z_1, z_2 of one of the dual solutions we will use are *asymmetric*, meaning that $z_1(x, y) \neq z_2(y, x)$. This comes in contrast with the approach in the main part of the thesis (see e.g. Sections 5.6 and 6.2), where for simplicity and generality, since our analysis had to be carried out for complicated settings of many items, symmetry was essential. This point further demonstrates the richness and the complexity of the problem of maximizing revenue in multidimensional settings: there might be many different dual solutions giving rise to the same value, both symmetric or asymmetric.

A.4.1 Full Bundling

Recall from (3.4) that the expected revenue in our case is

$$\lambda_1\lambda_2 \int_0^\infty \int_0^\infty u(x,y)\,(\lambda_1 x + \lambda_2 y - 3)\, e^{-(\lambda_1 x+\lambda_2 y)}\, dx\, dy$$

and using the change of variables $x = tz$ and $y = (1-t)z$ with $z \in \mathbb{R}_+$ and $t \in [0,1]$, having a Jacobian of z, this becomes

$$\lambda_1\lambda_2 \int_0^\infty z \int_0^1 u\,(tz,(1-t)z)\,[\lambda_1 tz + \lambda_2(1-t)z - 3]\, e^{-z(\lambda_1 t+\lambda_2(1-t))}\, dt\ dz.$$

By restricting our attention to functions u that are constant at every line of the form $x + y = z$, and defining $U(z) = u\,(tz,(1-t)z)$, which is constant for all $t \in [0,1]$, this becomes

$$\lambda_1\lambda_2 \int_0^\infty U(z) z \int_0^1 [\lambda_1 tz + \lambda_2(1-t)z - 3]\, e^{-z(\lambda_1 t+\lambda_2(1-t))}\, dt\ dz$$

which is equal to

$$\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2} \int_0^\infty U(z) \left[e^{-\lambda_1 z}(2-\lambda_1 z) - e^{-\lambda_2 z}(2-\lambda_2 z)\right] dz$$

and again integrating by parts, finally the expected revenue is

$$\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2} \int_0^\infty U'(z) \left[e^{-\lambda_2 z}\left(z-\frac{1}{\lambda_2}\right) - e^{-\lambda_1 z}\left(z-\frac{1}{\lambda_1}\right)\right] dz.$$

Selecting the full-bundling mechanism (from now one let's call it BUNDLE) with $u(x,y) = x + y - \zeta$ for $x + y \geq \zeta$ and $u(x,y) = 0$ otherwise, where $\zeta = \zeta(\lambda_1,\lambda_2)$ is the unique (remember that $\lambda_1 \neq \lambda_2$) root of the (strictly increasing) function

$$e^{-\lambda_2 z}\left(z-\frac{1}{\lambda_2}\right) - e^{-\lambda_1 z}\left(z-\frac{1}{\lambda_1}\right)$$

for $z \in \mathbb{R}_+$, i.e. having $U'(z) = 1$ for $z \geq \zeta$ and $U'(z) = 0$ otherwise, we get an expected revenue of

$$\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2} \int_{\zeta(\lambda_1,\lambda_2)}^\infty e^{-\lambda_2 z}\left(z-\frac{1}{\lambda_2}\right) - e^{-\lambda_1 z}\left(z-\frac{1}{\lambda_1}\right) dz = \zeta \frac{\lambda_1 e^{-\lambda_2\zeta} - \lambda_2 e^{-\lambda_1\zeta}}{\lambda_1-\lambda_2}. \tag{A.25}$$

You can see a graphical representation of the allocation space and the prices of BUNDLE in Figure A.4.

Since we are planning to use our duality-theory framework, and in particular weak duality (see Theorem 6.3), we want to find absolutely continuous functions z_1, z_2 :

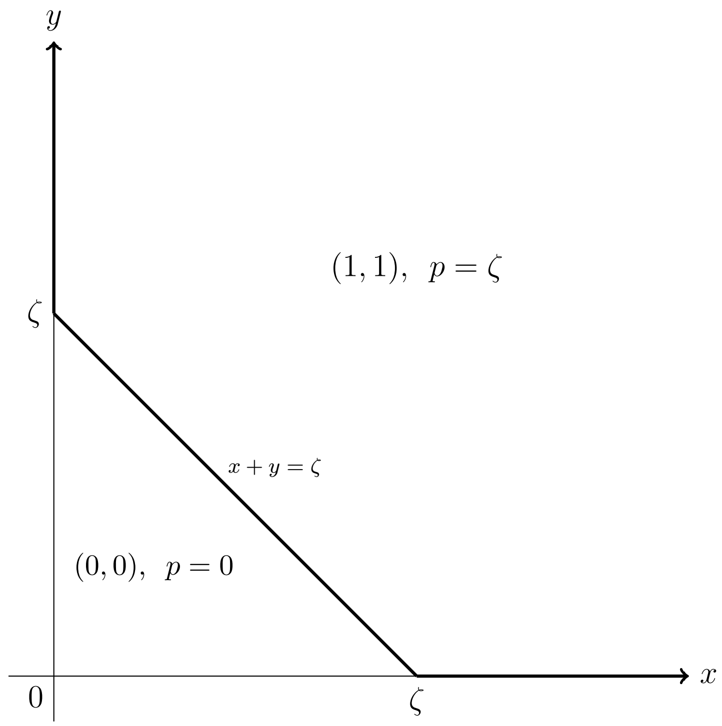


Figure A.4: Allocation function and payments of mechanism BUNDLE

$\mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ satisfying the following properties:

$$\frac{\partial z_1(x,y)}{\partial x} + \frac{\partial z_2(x,y)}{\partial y} \leq \lambda_1\lambda_2 \left[3 - (\lambda_1 x + \lambda_2 y)\right] e^{-(\lambda_1 x + \lambda_2 y)} \tag{A.26}$$

and

$$z_1(0,y) = z_2(x,0) = 0 \tag{A.27}$$

for all $x, y \in \mathbb{R}_+$, which give a dual value that can approximate well the expected revenue in (A.25).

We select our dual variables z_1, z_2 in the following way:

$$z_1(x,y) = \begin{cases} \lambda_2 \frac{\lambda_1 x}{\lambda_1 x + \lambda_2 y} \left[\lambda_1 x + \lambda_2 y - 1 - \frac{1}{\lambda_1 x + \lambda_2 y}\right] e^{-(\lambda_1 x + \lambda_2 y)}, & \lambda_1 x + \lambda_2 y \geq \phi, \\ 0, & \lambda_1 x + \lambda_2 y < \phi, \end{cases}$$
$$z_2(x,y) = \begin{cases} \lambda_1 \frac{\lambda_2 y}{\lambda_1 x + \lambda_2 y} \left[\lambda_1 x + \lambda_2 y - 1 - \frac{1}{\lambda_1 x + \lambda_2 y}\right] e^{-(\lambda_1 x + \lambda_2 y)}, & \lambda_1 x + \lambda_2 y \geq \phi, \\ 0, & \lambda_1 x + \lambda_2 y < \phi, \end{cases}$$

where ϕ here is the *golden ratio*, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$. Then, it is trivial to check that condition (A.27) is satisfied and also it is easy to see that z_1 and z_2 are nonnegative functions. For condition (A.26) first compute:

$$\frac{\partial z_1(x,y)}{\partial x} = \begin{cases} -\lambda_1\lambda_2 e^{-x\lambda_1-y\lambda_2} \frac{-x\lambda_1-x^2\lambda_1^2-2x^3\lambda_1^3+x^4\lambda_1^4+y\lambda_2-5x^2y\lambda_1^2\lambda_2+3x^3y\lambda_1^3\lambda_2+y^2\lambda_2^2-4xy^2\lambda_1\lambda_2^2+3x^2y^2\lambda_1^2\lambda_2^2-y^3\lambda_2^3+xy^3\lambda_1\lambda_2^3}{(x\lambda_1+y\lambda_2)^3}, & \lambda_1 x+\lambda_2 y>\phi, \\ 0, & \lambda_1 x+\lambda_2 y<\phi, \end{cases}$$

$$\frac{\partial z_2(x,y)}{\partial y} = \begin{cases} -\lambda_1\lambda_2 e^{-x\lambda_1-y\lambda_2} \frac{x\lambda_1+x^2\lambda_1^2-x^3\lambda_1^3-y\lambda_2-4x^2y\lambda_1^2\lambda_2+x^3y\lambda_1^3\lambda_2-y^2\lambda_2^2-5xy^2\lambda_1\lambda_2^2+3x^2y^2\lambda_1^2\lambda_2^2-2y^3\lambda_2^3+3xy^3\lambda_1\lambda_2^3+y^4\lambda_2^4}{(x\lambda_1+y\lambda_2)^3}, & \lambda_1 x+\lambda_2 y>\phi, \\ 0, & \lambda_1 x+\lambda_2 y<\phi, \end{cases}$$

and by summing up we satisfy condition (A.26) for all $(x,y)\in\mathbb{R}_+^2$, with equality in the subspace of $x\lambda_1+y\lambda_2\geq\phi$. The value of this feasible dual solution is

$$\begin{aligned}
\int_0^\infty\int_0^\infty (z_1(x,y)+z_2(x,y))\,dx\,dy &= \lambda_1\lambda_2 \int\limits_{\substack{x,y\geq 0\\ \lambda_1x+\lambda_2y\geq\phi}} (x+y)e^{-(\lambda_1x+\lambda_2y)}\left[1-\frac{1}{\lambda_1x+\lambda_2y}-\frac{1}{(\lambda_1x+\lambda_2y)^2}\right]dx\,dy \\
&= \int_\phi^\infty \left(z-1-\frac{1}{z}\right)e^{-z}\int_0^1 t\frac{z}{\lambda_1}+(1-t)\frac{z}{\lambda_2}\,dt\;\;dz \\
&= \int_\phi^\infty \left(z-1-\frac{1}{z}\right)e^{-z}\frac{1}{2}\left(\frac{z}{\lambda_1}+\frac{z}{\lambda_2}\right)\,dz \\
&= \frac{\lambda_1+\lambda_2}{2\lambda_1\lambda_2}\int_\phi^\infty \left(z^2-z-1\right)e^{-z}\,dz \qquad \text{(A.28)}
\end{aligned}$$

where for the third equality we used the change of variables $x=t\frac{z}{\lambda_1}$ and $y=(1-t)\frac{z}{\lambda_2}$ with $z\in\mathbb{R}_+$ and $t\in[0,1]$, having Jacobian equal to $\frac{z}{\lambda_1\lambda_2}$.

A.4.2 The Randomized Mechanism

In this section we present the almost optimal *randomized* mechanism for our case of two exponential goods. This mechanism is combination of the deterministic BUNDLE of the previous Appendix A.4.1 and a very simple randomized auction which has a menu-size [39] of just three: a full-bundling deterministic region (similar to BUNDLE), a single non-deterministic region with allocation probabilities 1 and $\frac{\lambda_2}{\lambda_1}$ for the two items (similar to PROPORTIONAL) and, of course, a "zero" region. Formally, our mechanism 3-RANDOM is the following:

Definition A.3 (3-RANDOM mechanism)**.** Mechanism 3-RANDOM allocates the items with probabilities

- $(0,0)$ if $\lambda_1x+\lambda_2y<2$ and $x+y<b$,
- $\left(1,\frac{\lambda_2}{\lambda_1}\right)$ for a price of $\frac{2}{\lambda_1}$ if $\lambda_1x+\lambda_2y\geq 2$ and $y\leq c$, and
- $(1,1)$ for a price of b if $x+y\geq b$ and $y>c$,

respectively, where $b=\frac{1-W\left(-\frac{\lambda_2}{e\lambda_1}\right)}{\lambda_2}$ and $c=\frac{\lambda_1 b-2}{\lambda_1-\lambda_2}$. Here W is the Lambert function (see also Footnote 4).

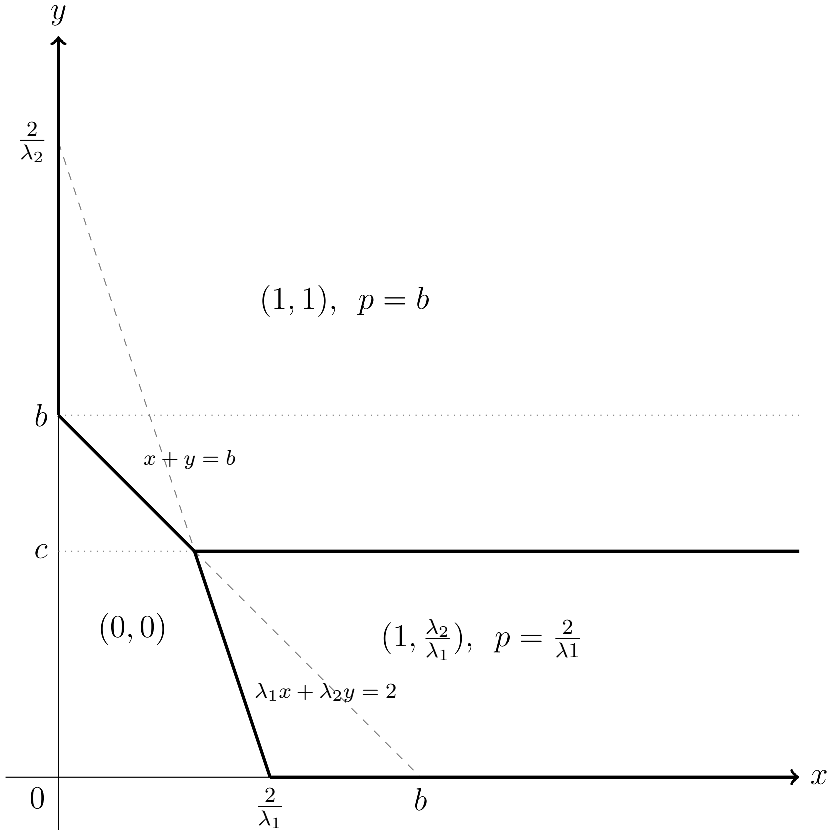


Figure A.5: The allocation function and payments of mechanism 3-Random, for two exponential goods. Here $b = \frac{1-W\left(-\frac{\lambda_2}{e\lambda_1}\right)}{\lambda_2}$ and $c = \frac{\lambda_1 b - 2}{\lambda_1 - \lambda_2}$.

First of all, notice that 3-Random is a *truthful* mechanism, since it is produced by the convex utility function

$$u(x,y) = \max\left\{0, x + \frac{\lambda_2}{\lambda_1}y - \frac{2}{\lambda_1}, x + y - b\right\},$$

and also it is well defined for $\frac{\lambda_2}{\lambda_1} \leq \ell_1$, where $\ell_1 \approx 0.678$ is the (unique) root of the equation

$$W(-e^{-1}x) = 1 - 2x$$

for $x \in (0,1)$, since $c < b < \frac{2}{\lambda_2}$ for all $\lambda_1 > \lambda_2 > 0$ but also $c \geq 0$ for $\frac{\lambda_2}{\lambda_1} \leq \ell_1$. A graphical representation of the allocation space of the mechanism and its prices is given in Figure A.5.

Now, we are ready to give our complete selling randomized mechanism for two non-i.i.d. exponential goods:

Definition A.4 (Mixed auction)**.**

- If $\frac{\lambda_1}{\lambda_2} \in (0, \ell_1]$ then run mechanism 3-Random, and
- If $\frac{\lambda_1}{\lambda_2} \in (\ell_1, 1)$ then run mechanism Bundle,

where $\ell_1 \approx 0.678$ is defined above.

The expected revenue of BUNDLE was analyzed in Appendix A.4.1 and is given by expression (A.25). Let's now focus on analyzing 3-RANDOM. By Definition A.3 we have that the expected revenue is:

$$\int_0^c \int_{\frac{2-\lambda_2 y}{\lambda_1}}^{\infty} \frac{2}{\lambda_1} \cdot \lambda_1 \lambda_2 e^{-\lambda_1 x - \lambda_2 y}\, dx\, dy + \int_c^b \int_{b-y}^{\infty} b \cdot \lambda_1 \lambda_2 e^{-\lambda_1 x - \lambda_2 y}\, dx\, dy + \int_b^{\infty} \int_0^{\infty} b \cdot \lambda_1 \lambda_2 e^{-\lambda_1 x - \lambda_2 y}\, dx\, dy$$

which equals

$$\frac{2c\lambda_2}{e^2 \lambda_1} + \frac{b\lambda_2 \left(e^{-b\lambda_2} - e^{-b\lambda_1 + c\lambda_1 - c\lambda_2}\right)}{\lambda_1 - \lambda_2} + be^{-b\lambda_2}$$

and by using the facts that $c = \frac{\lambda_1 b - 2}{\lambda_1 - \lambda_2}$ and $b = \frac{1 - W\left(-\frac{\lambda_2}{e\lambda_1}\right)}{\lambda_2}$, we get that the expected revenue of 3-RANDOM is given by

$$\frac{\lambda_2 e^{-2}}{\lambda_1(\lambda_1 - \lambda_2)} \left[\frac{\lambda_1 \lambda_2 b^2}{\lambda_2 b - 1} - 4\right]. \tag{A.29}$$

We will now utilize weak duality to show MIXED mechanism has *almost optimal* revenue. We need to find absolutely continuous functions $z_1, z_2 : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ satisfying the properties (A.26) and (A.27) for all $x, y \in \mathbb{R}_+$. Then, the quantity $\int_0^\infty \int_0^\infty (z_1(x,y) + z_2(x,y))\, dx\, dy$ will bound MIXED's (and in fact *every* mechanism's for our problem) revenue from *above*.

The construction of the dual variables presented below is for $\frac{\lambda_2}{\lambda_1} \leq \ell_2 \equiv 1 + \frac{1}{2} W\left(-2e^{-2}\right) \approx 0.797$. Otherwise, we define the duals as presented in Appendix A.4.1. First let

$$z_1(x,y) = \begin{cases} \lambda_1 \lambda_2 \left(x + \frac{\lambda_2}{\lambda_1} y - \frac{2}{\lambda_1}\right) e^{-\lambda_1 x - \lambda_2 y}, & \lambda_1 x + \lambda_2 y \geq 2 \ \wedge \ 0 \leq y \leq \tilde{c}, \\ \lambda_1 \lambda_2 \left(x + y - \tilde{b}\right) e^{-\lambda_1 x - \lambda_2 y}, & x + y \geq \tilde{b} \ \wedge \ \tilde{c} < y \leq \tilde{b}, \\ \lambda_1 \lambda_2 x e^{-\lambda_1 x - \lambda_2 y}, & x \geq 0 \ \wedge \ y > \tilde{b}, \\ 0, & \text{otherwise}, \end{cases}$$

for all $x, y \in \mathbb{R}_+$, where we pick parameters $\tilde{b}$ and $\tilde{c}$ to be

$$\tilde{b} = \frac{2\lambda_2 + (\lambda_1 - \lambda_2)\log\left(1 - \frac{\lambda_2}{\lambda_1}\right)}{\lambda_2^2} = \frac{1}{\lambda_2}\left(2 - \lambda \log \lambda\right) \tag{A.30}$$

$$\tilde{c} = \frac{\lambda_1 \tilde{b} - 2}{\lambda_1 - \lambda_2} = \frac{2\lambda_2 + \lambda_1 \log\left(1 - \frac{\lambda_2}{\lambda_1}\right)}{\lambda_2^2} = \frac{1}{\lambda_2}\left(2 + \frac{\log \lambda}{1 - \lambda}\right), \tag{A.31}$$

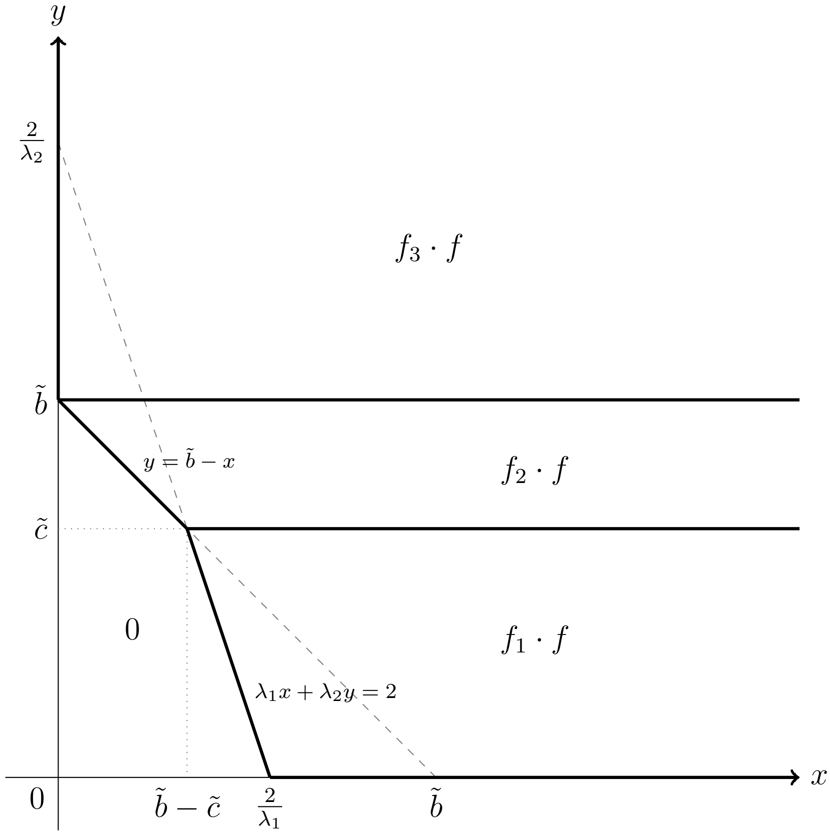


Figure A.6: The critical regions of the feasible dual solution z_1

where $\lambda = 1 - \frac{\lambda_1}{\lambda_2} \in (0, 1)$. We can write this more compactly as

$$z_1(x,y) = \begin{cases} h_1(x,y)h(x,y), & h_1(x,y) \geq 0 \ \wedge \ 0 \leq y \leq \tilde{c}, \\ h_2(x,y)h(x,y), & h_2(x,y) \geq 0 \ \wedge \ \tilde{c} < y \leq \tilde{b}, \\ h_3(x,y)h(x,y), & h_3(x,y) \geq 0 \ \wedge \ y > \tilde{b}, \\ 0, & \text{otherwise}, \end{cases}$$

where $h(x,y) = \lambda_1\lambda_2 e^{-\lambda_1 x - \lambda_2 y}$, $h_1(x,y) = x + \frac{\lambda_2}{\lambda_1}y - \frac{2}{\lambda_1}$, $h_2(x,y) = x + y - b$ and $h_3(x,y) = x$. Notice that z_1 is a well defined, absolutely continuous, nonnegative function in $\mathbb{R}_+^2$ since $\tilde{c} < \tilde{b} < \frac{2}{\lambda_2}$ for all $\lambda_1 > \lambda_2 > 0$ but also $\tilde{c} \geq 0$ for $\frac{\lambda_2}{\lambda_1} \leq \ell_2$. In addition, it is easy to see that $h_1(x,\tilde{c}) = h_2(x,\tilde{c})$ and $h_2(x,\tilde{b}) = h_3(x,\tilde{b})$ for all $x \in \mathbb{R}_+$. An illustration of the critical regions of z_1 and its values is given in Figure A.6.

The critical derivative of function z_1 is

$$\frac{\partial z_1(x,y)}{\partial x} = \begin{cases} (3 - \lambda_1 x - \lambda_2 y)h(x,y), & h_1(x,y) \geq 0 \ \wedge \ 0 \leq y \leq \tilde{c}, \\ (1 + \lambda_1\tilde{b} - \lambda_1 x - \lambda_1 y)h(x,y), & h_2(x,y) \geq 0 \ \wedge \ \tilde{c} < y \leq \tilde{b}, \\ (1 - \lambda_1 x)h(x,y), & h_3(x,y) \geq 0 \ \wedge \ y > \tilde{b}, \\ 0, & \text{otherwise}, \end{cases} \tag{A.32}$$

Now we move on to defining z_2. As we discussed in the introduction of this section, this time this is going to be non-symmetric with respect to the other dual variable z_1:

$$z_2(x,y) = \begin{cases} \lambda_1 e^{-\lambda_1 x}\left[\frac{\lambda_1}{\lambda_2}e^{-\lambda_2\tilde{b}} - \frac{\lambda_1-\lambda_2}{\lambda_2}e^{-\lambda_2(\tilde{b}-x)} + e^{-\lambda_2 y}(\lambda_1 x + \lambda_2 y - 2)\right], & x \le \tilde{b}-\tilde{c} \wedge \psi(x) \le y \le \tilde{b}-x, \\ \lambda_1 e^{-\lambda_1 x}\left[\frac{\lambda_1}{\lambda_2}e^{-\lambda_2\tilde{b}} + e^{-\lambda_2 y}\left(\lambda_1\tilde{b} - \frac{\lambda_1+\lambda_2}{\lambda_2} - (\lambda_1-\lambda_2)y\right)\right], & x \le \tilde{b}-\tilde{c} \wedge \tilde{b}-x < y \le \tilde{b}, \\ \lambda_1 e^{-\lambda_1 x}\left[\frac{\lambda_1-\lambda_2}{\lambda_2}e^{-\lambda_2\frac{\lambda_1\tilde{b}-2}{\lambda_1-\lambda_2}} + e^{-\lambda_2 y}\left(\lambda_1(\tilde{b}-1) - 1 - y(\lambda_1-\lambda_2)\right)\right], & x > \tilde{b}-\tilde{c} \wedge \tilde{c} \le y \le \tilde{b}, \\ \lambda_1(\lambda_2 y - 1)e^{-\lambda_1 x - \lambda_2 y}, & y \ge \tilde{b}, \\ 0, & \text{otherwise,} \end{cases}$$

for all $x, y \in \mathbb{R}_+$, where

$$\psi(x) = \frac{2}{\lambda_2} - \frac{\lambda_1}{\lambda_2}x - \frac{1}{\lambda 2}W\left(\frac{e^{-\lambda_1 x - \lambda_2\tilde{b}+2}\left(\lambda_1 - (\lambda_1-\lambda_2)e^{\lambda_2 x}\right)}{\lambda_2}\right) \tag{A.33}$$

and parameters $\tilde{b}$ and $\tilde{c}$ are as before. More compactly,

$$z_2(x,y) = \begin{cases} g_1(x,y)g(x,y) & x \le \tilde{b}-\tilde{c} \wedge \psi(x) \le y \le \tilde{b}-x, \\ g_2(x,y)g(x,y), & x \le \tilde{b}-\tilde{c} \wedge \tilde{b}-x < y \le \tilde{b}, \\ g_3(x,y)g(x,y), & x > \tilde{b}-\tilde{c} \wedge \tilde{c} \le y \le \tilde{b}, \\ g_4(x,y)g(x,y) & y \ge \tilde{b}, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} g(x,y) &= \lambda_1 e^{-\lambda_1 x} \\ g_1(x,y) &= \frac{\lambda_1}{\lambda_2}e^{-\lambda_2\tilde{b}} - \frac{\lambda_1-\lambda_2}{\lambda_2}e^{-\lambda_2(\tilde{b}-x)} + e^{-\lambda_2 y}(\lambda_1 x + \lambda_2 y - 2) \\ g_2(x,y) &= \frac{\lambda_1}{\lambda_2}e^{-\lambda_2\tilde{b}} + e^{-\lambda_2 y}\left(\lambda_1\tilde{b} - \frac{\lambda_1+\lambda_2}{\lambda_2} - (\lambda_1-\lambda_2)y\right) \\ g_3(x,y) &= \frac{\lambda_1-\lambda_2}{\lambda_2}e^{-\lambda_2\frac{\lambda_1\tilde{b}-2}{\lambda_1-\lambda_2}} + e^{-\lambda_2 y}\left(\lambda_1(\tilde{b}-1) - 1 - y(\lambda_1-\lambda_2)\right) \\ g_4(x,y) &= (\lambda_2 y - 1)e^{-\lambda_2 y}. \end{aligned}$$

Notice that, z_2 is also a well defined, nonnegative, absolutely continuous function in $\mathbb{R}^2_+$, based on the above analysis after the definition of z_1, and also because $g_1(x, b-x) = g_2(x, b-x)$ and $g_2(x,\tilde{b}) = g_3(x,\tilde{b}) = g_4(x,\tilde{b})$ for all $x \in \mathbb{R}_+$. Finally, it is also critical to point out that $0 < \psi(x) \le \tilde{b} - x$ for all $x \in [0, \tilde{b}-\tilde{c}]$ with $\psi(\tilde{b}-\tilde{c}) = \tilde{c}$. An illustration of the critical regions of z_2 and its values is given in Figure A.7. Notice the extra region between curves $y = \tilde{b} - x$ and $y = \psi(x)$ drawn with blue colour, that differentiates the "projection" of z_2 from that of z_1.

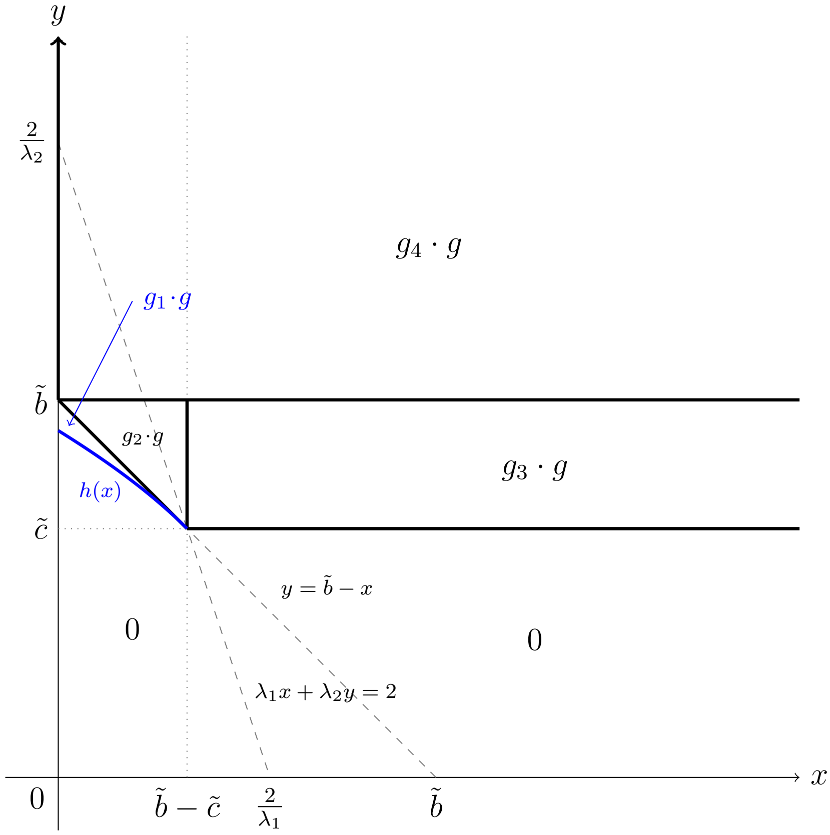


Figure A.7: The critical regions of the feasible dual solution z_2. Notice the asymmetry with respect to the other part of the solution, namely z_1 in Figure A.6.

The derivative of z_2 is

$$
\frac{\partial z_2(x,y)}{\partial y} = \begin{cases} \lambda_2(3 - \lambda_1 x - \lambda_2 y)e^{-\lambda_2 y} g(x,y), & x \le \tilde{b} - \tilde{c} \ \wedge \ \psi(x) \le y \le \tilde{b} - x, \\ \lambda_2(2 - \tilde{b}\lambda_1 + (\lambda_1 - \lambda_2)y)e^{-\lambda_2 y} g(x,y), & x \le \tilde{b} - \tilde{c} \ \wedge \ \tilde{b} - x < y \le \tilde{b}, \\ \lambda_2(2 - \tilde{b}\lambda_1 + (\lambda_1 - \lambda_2)y)e^{-\lambda_2 y} g(x,y), & x > \tilde{b} - \tilde{c} \ \wedge \ \tilde{c} \le y \le \tilde{b}, \\ \lambda_2(2 - \lambda_2 y)e^{-\lambda_2 y} g(x,y), & y \ge \tilde{b}, \\ 0, & \text{otherwise,} \end{cases}
$$

$$
= \begin{cases} (3 - \lambda_1 x - \lambda_2 y)h(x,y), & x \le \tilde{b} - \tilde{c} \ \wedge \ \psi(x) \le y \le \tilde{b} - x, \\ (2 - \tilde{b}\lambda_1 + (\lambda_1 - \lambda_2)y)h(x,y), & h_2(x,y) \ge 0 \wedge \ \tilde{c} \le y \le \tilde{b}, \\ (2 - \lambda_2 y)h(x,y), & y \ge \tilde{b}, \\ 0, & \text{otherwise.} \end{cases} \tag{A.34}
$$

From equations (A.32) and (A.34) one can see that condition (A.26) is indeed satisfied. Now it remains to upper-bound the integral $\int_{\mathbb{R}^2_+} z_1 + z_2$.

First, we compute

$$\begin{aligned}\int_0^\infty \int_0^\infty z_1(x,y)\,dx\,dy &= \int_0^{\tilde{c}} \int_{\frac{2-\lambda_2 y}{\lambda_1}}^\infty h_1(x,y)h(x,y)\,dx\,dy + \int_{\tilde{c}}^{\tilde{b}} \int_{\tilde{b}-y}^\infty h_2(x,y)h(x,y)\,dx\,dy \\ &\quad + \int_{\tilde{b}}^\infty \int_0^\infty h_3(x,y)h(x,y)\,dx\,dy \\ &= \frac{\lambda_2\tilde{c}}{\lambda_1 e^2} + \frac{\lambda_2}{\lambda_1(\lambda_1-\lambda_2)}\left(e^{-\lambda_2\tilde{b}} - e^{-\lambda_1\tilde{b}+\tilde{c}(\lambda_1-\lambda_2)}\right) + \frac{e^{-\lambda_2\tilde{b}}}{\lambda_1} \\ &= \frac{\lambda_1 e^{-\lambda_2\tilde{b}-2} + (\lambda_1\tilde{b}-3)\lambda_2}{\lambda_1(\lambda_1-\lambda_2)e^2}, \qquad \text{using (A.31).}\end{aligned}$$

Next,

$$\begin{aligned}\int_0^{\tilde{b}-\tilde{c}} \int_{\psi(x)}^{\tilde{b}-x} g_1(x,y)g(x,y)\,dy\,dx &= \frac{\lambda_1}{\lambda_2}\int_0^{\tilde{b}-\tilde{c}} \left((\lambda_1-\lambda_2)\psi(x)+1-\lambda_1\tilde{b}\right)e^{-\lambda_2\tilde{b}-(\lambda_1-\lambda_2)x} - \lambda_1(h(x)+x-b)e^{-\lambda_2\tilde{b}-\lambda_1 x} \\ &\quad + (\lambda_2\psi(x)+\lambda_1 x-1)e^{-\lambda_2\psi(x)-\lambda_1 x}\,dx \\ &\equiv i_1(\lambda_1,\lambda_2) \\ \int_0^{\tilde{b}-\tilde{c}} \int_{\tilde{b}-x}^{\tilde{b}} g_2(x,y)g(x,y)\,dy\,dx &= \frac{1}{\lambda_2^2(\lambda_1-\lambda_2)}\Big[\lambda_1(2\lambda_1-\lambda_2)e^{-2} \\ &\quad + \lambda_2(\lambda_2-1)e^{-\lambda_2\tilde{b}} + \left(\lambda_2^2(2\lambda_1\tilde{b}+1-\tilde{b}) - \lambda_1(2\lambda_1+\lambda_2)\right)e^{\frac{-2\lambda_1+\tilde{b}\lambda_2^2}{\lambda_1-\lambda_2}}\Big] \\ &= \frac{2\lambda_2^2\tilde{b} - \lambda_1(\tilde{b}+2\lambda_2\tilde{b}-2) + \lambda_2(\tilde{b}-1)}{\lambda_1(\lambda_1-\lambda_2)}e^{-2} + \frac{\lambda_2-1}{\lambda_2(\lambda_1-\lambda_2)}e^{-\lambda_2\tilde{b}} \quad \text{by (A.30)} \\ \int_{\tilde{c}}^{\tilde{b}} \int_{\tilde{b}-\tilde{c}}^\infty g_3(x,y)g(x,y)\,dx\,dy &= \frac{e^{-2}}{\lambda_2^2}\left[(2\lambda_1-\lambda_2^2\tilde{b})e^{-\frac{\lambda_2(2-\lambda_2\tilde{b})}{\lambda_1-\lambda_2}} - 2\lambda_1 + 4\lambda_2 - \lambda_2^2\tilde{b}\right] \\ &= \frac{2\lambda_1 - 2\lambda_1\lambda_2\tilde{b} + \lambda_2^2\tilde{b}}{\lambda_1\lambda_2}e^{-2} \quad \text{by (A.30)} \\ \int_{\tilde{b}}^\infty \int_0^\infty g_4(x,y)g(x,y)\,dx\,dy &= \tilde{b}e^{-\lambda_2\tilde{b}}.\end{aligned}$$

Combining all these we finally get that:

$$\int_0^\infty \int_0^\infty z_1(x,y) + z_2(x,y)\,dx\,dy = i_1(\lambda_1,\lambda_2) + i_2(\lambda_1,\lambda_2), \tag{A.35}$$

where

$$i_2(\lambda_1,\lambda_2) = \frac{\lambda_2\tilde{b}\left((\lambda_2-1)(\lambda_1-\lambda_2)+\lambda_1^2\right) + 2\lambda_1^2 - \lambda_2(3\lambda_1+\lambda_2)}{\lambda_1\lambda_2(\lambda_1-\lambda_2)}e^{-2} + \left(b + \frac{\lambda_1+\lambda_2-1}{\lambda_2(\lambda_1-\lambda_2)}\right)e^{-\lambda_2\tilde{b}}.$$

Summarizing, from equations (A.25) and (A.29) we have that the expected revenue of the Mixed mechanism is

$$\mathcal{R}_{\text{Mixed}}(\lambda_1,\lambda_2) = \begin{cases} \frac{\lambda_2 e^{-2}}{\lambda_1(\lambda_1-\lambda_2)}\left(\frac{\lambda_1\lambda_2 b^2}{\lambda_2 b-1} - 4\right), & 0 < \frac{\lambda_2}{\lambda_1} \le \ell_1, \\ \zeta\frac{\lambda_1 e^{-\lambda_2\zeta} - \lambda_2 e^{-\lambda_1\zeta}}{\lambda_1-\lambda_2}, & \ell_1 < \frac{\lambda_2}{\lambda_1} < 1, \end{cases} \tag{A.36}$$

where $\ell_1 \approx 0.678101$ is the (unique) root of the equation $W(-e^{-1}x) = 1-2x$ in $(0,1)$,

$b = \frac{1-W\left(-\frac{\lambda_2}{e\lambda_1}\right)}{\lambda_2}$ and ζ is the (unique) root of function $e^{-\lambda_2 z}\left(z - \frac{1}{\lambda_2}\right) - e^{-\lambda_1 z}\left(z - \frac{1}{\lambda_1}\right)$ in $\mathbb{R}_+$. On the other hand, by combining our dual constructions of Appendices A.4.1 and A.4.2 we get that the optimal revenue is upper bounded by

$$\mathcal{R}_{\mathrm{OPT}}(\lambda_1, \lambda_2) \leq \begin{cases} \min\left(i_1(\lambda_1,\lambda_2) + i_2(\lambda_1,\lambda_2), \frac{\lambda_1+\lambda_2}{2\lambda_1\lambda_2}\int_\phi^\infty \left(z^2 - z - 1\right) e^{-z}\, dz\right), & 0 < \frac{\lambda_2}{\lambda_1} \leq \ell_2, \\ \frac{\lambda_1+\lambda_2}{2\lambda_1\lambda_2}\int_\phi^\infty \left(z^2 - z - 1\right) e^{-z}\, dz, & \ell_2 < \frac{\lambda_2}{\lambda_1} < 1, \end{cases} \tag{A.37}$$

by equations (A.35) and (A.28). The following theorem establishes that the ratio $\frac{\mathcal{R}_{\mathrm{DUAL}}(\lambda_1,\lambda_2)}{\mathcal{R}_{\mathrm{MIXED}}(\lambda_1,\lambda_2)}$ depends only on the value of the ratio $\frac{\lambda_2}{\lambda_2}$ and not at the independent values of λ_1 and λ_2:

Theorem A.4. *Let $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in \mathbb{R}_+$. Then*

$$\frac{\lambda_2'}{\lambda_1'} = \frac{\lambda_2}{\lambda_1} \quad \Longrightarrow \quad \frac{\mathcal{R}_{\mathrm{DUAL}}(\lambda_1', \lambda_2')}{\mathcal{R}_{\mathrm{MIXED}}(\lambda_1', \lambda_2')} = \frac{\mathcal{R}_{\mathrm{DUAL}}(\lambda_1, \lambda_2)}{\mathcal{R}_{\mathrm{MIXED}}(\lambda_1, \lambda_2)}.$$

Proof. See the following **??** A.4.2.1. □

So, we define this ratio

$$\rho\left(\frac{\lambda_2}{\lambda_1}\right) = \frac{\mathcal{R}_{\mathrm{DUAL}}(\lambda_1, \lambda_2)}{\mathcal{R}_{\mathrm{MIXED}}(\lambda_1, \lambda_2)}$$

and of course, from our duality framework, it is obvious that this is an upper bound on the approximation ratio of the revenue of mechanism Mixed. You can see a plot of $\rho(\lambda)$ for all possible values of $\lambda \in (0, 1)$ in Figure A.8. Thus, the following result is immediate:

Theorem A.5. *The* Mixed *mechanism for the single-player two-items model with non-identical exponentially distributed valuation priors is ρ-approximate, with $\rho <$ 1.0033.*

A.4.2.1 The approximation ratio depends only on the ratio $\frac{\lambda_2}{\lambda_1}$

Lemma A.1. *Let $\lambda_1, \lambda_2, \lambda_1', \lambda_2', c \in \mathbb{R}_{>0}$ such that $\lambda_1' = c\lambda_1$ and $\lambda_2' = c\lambda_2$. Then,*

$$\mathcal{R}_{\mathrm{MIXED}}(\lambda_1', \lambda_2') = \frac{1}{c}\mathcal{R}_{\mathrm{MIXED}}(\lambda_1, \lambda_2).$$

Proof. First, from the way we defined b in Definition A.3, it is straightforward to see that $b' = \frac{b}{c}$. Also, for every real variable $z \in \mathbb{R}_+$

$$e^{-\lambda_2 z}\left(z - \frac{1}{\lambda_2}\right) - e^{-\lambda_1 z}\left(z - \frac{1}{\lambda_1}\right) = 0 \Longleftrightarrow e^{-c\lambda_2 \frac{z}{c}}\left(\frac{z}{c} - \frac{1}{c\lambda_2}\right) - e^{-c\lambda_1 \frac{z}{c}}\left(\frac{z}{c} - \frac{1}{c\lambda_1}\right) = 0$$
$$\Longleftrightarrow e^{-\lambda_2' \frac{z}{c}}\left(\frac{z}{c} - \frac{1}{\lambda_2'}\right) - e^{-\lambda_1' \frac{z}{c}}\left(\frac{z}{c} - \frac{1}{\lambda_1'}\right) = 0,$$

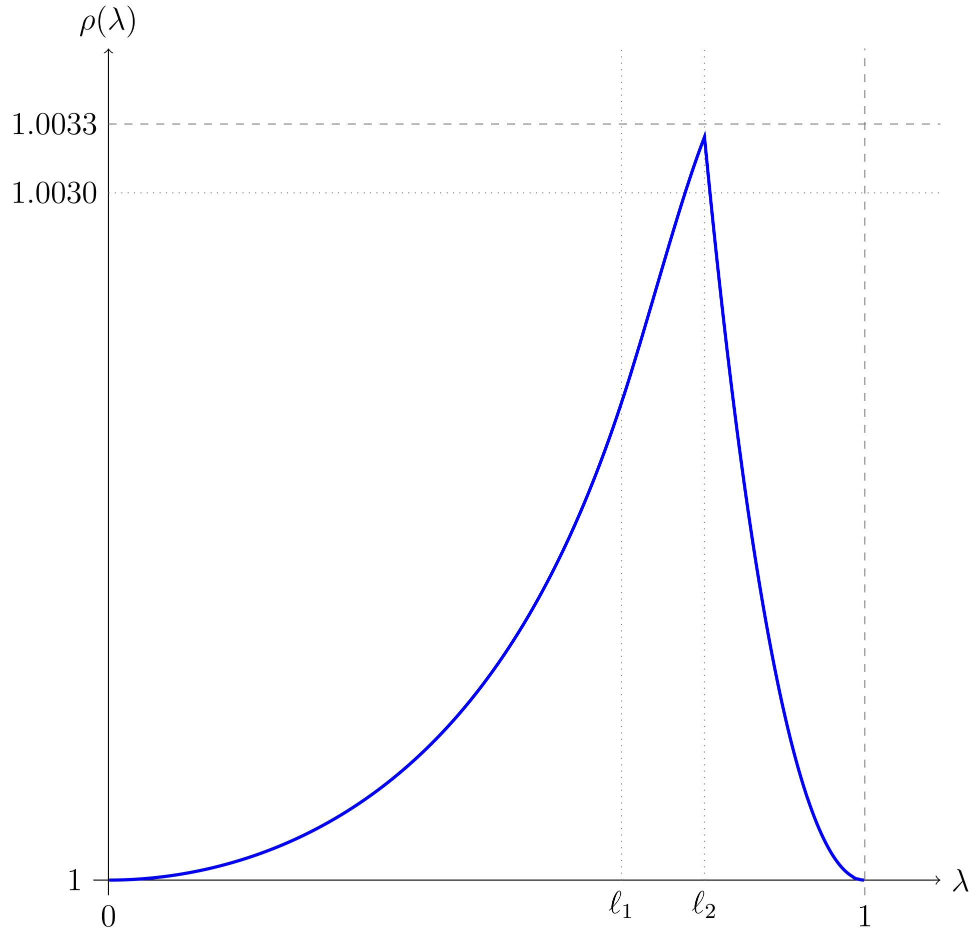


Figure A.8: Plot of $\rho(\lambda)$ with respect to the ratio λ

so in the way we defined ζ in Appendix A.4.1 it is again now easy to see that $\zeta' = \frac{\zeta}{c}$.

By the definition of MIXED mechanism we have two cases from (A.36) to study in order to establish the lemma:

- Case 1: $0 < \frac{\lambda_1}{\lambda_2} = \frac{\lambda_1'}{\lambda_2'} < \ell_1$. Then

$$\mathcal{R}_{\text{MIXED}}(\lambda_1', \lambda_2') = \frac{\lambda_2' e^{-2}}{\lambda_1'(\lambda_1' - \lambda_2')}\left(\frac{\lambda_1'\lambda_2' b'^2}{\lambda_2' b' - 1} - 4\right) = \frac{c\lambda_2 e^{-2}}{c\lambda_1(c\lambda_1 - c\lambda_2)}\left(\frac{c\lambda_1 c\lambda_2 \frac{b^2}{c^2}}{c\lambda_2 \frac{b}{c} - 1} - 4\right) = \frac{1}{c}\mathcal{R}_{\text{MIXED}}(\lambda_1, \lambda_2).$$

- Case 2: $\ell_1 < \frac{\lambda_1}{\lambda_2} = \frac{\lambda_1'}{\lambda_2'} < 1$. Then

$$\mathcal{R}_{\text{MIXED}}(\lambda_1', \lambda_2') = \zeta' \frac{\lambda_1' e^{-\lambda_2'\zeta'} - \lambda_2' e^{-\lambda_1'\zeta'}}{\lambda_1' - \lambda_2'} = \frac{\zeta}{c}\frac{c\lambda_1 e^{-c\lambda_2\frac{\zeta}{c}} - c\lambda_2 e^{-c\lambda_1\frac{\zeta}{c}}}{c\lambda_1 - c\lambda_2} = \frac{1}{c}\mathcal{R}_{\text{MIXED}}(\lambda_1, \lambda_2).$$

□

Lemma A.2. *Let $\lambda_1, \lambda_2, \lambda_1', \lambda_2', c \in \mathbb{R}_{>0}$ such that $\lambda_1' = c\lambda_1$ and $\lambda_2' = c\lambda_2$. Then,*

$$\mathcal{R}_{\text{OPT}}(\lambda_1', \lambda_2') = \frac{1}{c}\mathcal{R}_{\text{OPT}}(\lambda_1, \lambda_2).$$

Proof. From the way we defined $\tilde{b}$ in (A.30), it is easy to see that $\tilde{b}' = \frac{\tilde{b}}{c}$. Then, it also not difficult to see from (A.33) that

$$\psi'\left(\frac{x}{c}\right) = \frac{1}{c}\psi(x).$$

Now, taking the above into consideration it is a matter of elementary calculations to look at the definitions in page 154 and check that

$$i_1(\lambda_1', \lambda_2') + i_2(\lambda_1', \lambda_2') = \frac{1}{c}\left(i_1(\lambda_1, \lambda_2) + i_2(\lambda_1, \lambda_2)\right),$$

and it is also trivial to see that

$$\frac{\lambda_1' + \lambda_2'}{2\lambda_1'\lambda_2'} \int_\phi^\infty \left(z^2 - z - 1\right) e^{-z}\, dz = \frac{1}{c}\frac{\lambda_1 + \lambda_2}{2\lambda_1\lambda_2} \int_\phi^\infty \left(z^2 - z - 1\right) e^{-z}\, dz.$$

So from equation (A.37) we get the desired

$$\mathcal{R}_{\mathrm{OPT}}(\lambda_1', \lambda_2') = \frac{1}{c}\mathcal{R}_{\mathrm{OPT}}(\lambda_1, \lambda_2).$$

□

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